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MALLIAVIN CALCULUS

WITH APPLICATIONS TO STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

Marta Sanz-Solé

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Introduction

Malliavin calculus is a stochastic calculus of variations on the Wiener space. Its foundations were set in the 1970's, mainly in the seminal work [33], in order to study the existence and smoothness of density for the *probability laws* of random vectors. For diffusion processes this problem can be approached by applying *Hörmander's theorem* on *hypoelliptic differential operators* in square form to *Kolmogorov's equation* (see ref. [18]). Thus, in its very first application, Malliavin calculus provides a probabilistic proof of the above mentioned Hörmander's theorem. Actually the first developments of the theory consist of a probabilistic theory for second order elliptic and parabolic stochastic partial differential equations with the broad contributions by Kusuoka and Stroock, Ikeda and Watanabe, Bell and Mohammed, among others. As a sample of references and without aiming to be complete, we mention [6], [19] and [28]–[30]. Further developments in the analysis on the *Wiener space* led to contributions in many areas of probability theory. Let us mention for instance the theory of *Dirichlet forms* and applications to error calculus (see the monograph [8]) and the *anticipating stochastic calculus* with respect to Gaussian processes (refs. [45], [46] and [58]). At a more applied level, Malliavin calculus is used in probabilistic numerical methods in financial mathematics. Many problems in probability theory have been and are being successfully approached with tools from Malliavin calculus. Here are some samples together with basic references:

1) Small perturbations of *stochastic dynamical systems* (refs. [10], [11], [26], [27] and [31])
2) Weak results on existence of solutions for *parabolic stochastic partial differential equations* and numerical approximations (refs. [1] and [2])
3) Time reversal for finite and *infinite dimensional stochastic differential equations* (refs. [38] and [39])
4) Transformation of measure on the Wiener space (ref. [66])
5) Extension of Itô's formulae (refs. [5] and [41])
6) *Potential theory* (ref. [15])

The aim of this book is to present applications of Malliavin calculus to the analysis of probability laws of solutions of stochastic partial differential equations driven by Gaussian noises which are white in time and coloured in space, in a comprehensive way. The first five chapters are devoted to the introduction to the calculus itself based on a general *Gaussian space*, going from the simple finite-dimensional setting to the infinite-dimensional one. The last three chapters are devoted to the applications to *stochastic partial differential equations* based on recent research. Each chapter ends with some comments concerning the origin of the work developed within and its references. Throughout the paper, we denote by C a real positive constant which can vary from a line to another.

The notes were written on the occasion of a visit to the *Institut de Mathématiques* at the Swiss Federal Institute of Technology in Lausanne in Fall 2003. I take this opportunity to thank the institution for the invitation. I am deeply indebted to Professor Robert Dalang for providing me a very inspiring scientific atmosphere, for his valuable collaboration and for the meticulous and critical reading of a version of the manuscript. My thanks are also due to Lluís Quer-Sardanyons for the careful reading of its first version and to those who attended the course for their enthousiasm, interest and remarks.

Contents

Chapter 1

Integration by Parts and Absolute Continuity of Probability Laws

In this chapter, we give some general results on the existence of *density* for probability laws and properties of these densities. There are different approaches depending on whether one wishes to compute the densities —and even their derivatives— or not. The criteria proved by Malliavin in reference [33] establish existence and smoothness of the density (see Proposition 1.2). The approach by Watanabe (ref. [68]) yields —under stronger assumptions— a description of the densities and their derivatives.

We present here a review of these results, putting more emphasis on the second approach.

Let us first introduce some notation. Derivative multiindices are denoted by $\alpha = (\alpha_1, \dots, \alpha_r) \in \{1, \dots, n\}^r$. Set $|\alpha| = \sum_{i=1}^r \alpha_i$. For any differentiable real valued function φ defined on $\mathbb{R}^n$, we denote by $\partial_\alpha \varphi$ the partial derivative $\partial^{|\alpha|}_{\alpha_1,\dots,\alpha_r}\varphi$. If $|\alpha| = 0$, $\partial_\alpha\varphi = \varphi$, by convention.

Definition 1.1 Let F be a $\mathbb{R}^n$-valued random vector, $F = (F_1, \dots, F_n)$, and G be an integrable random variable defined on some probability space $(\Omega, \mathcal{F}, P)$. Let α be a multiindex. The pair F, G satisfies an *integration by parts formula* of degree α if there exists a random variable $H_\alpha(F,G) \in L^1(\Omega)$ such that

$$E\big((\partial_\alpha\varphi)(F)G\big) = E\big(\varphi(F)H_\alpha(F,G)\big), \tag{1.1}$$

for any $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$.

The property expressed in (1.1) is recursive in the following sense. Let $\alpha = (\beta, \gamma)$, with $\beta = (\beta_1, \dots, \beta_a)$, $\gamma = (\gamma_1, \dots, \gamma_b)$. Then

$$\begin{aligned} E\big((\partial_\alpha \varphi)(F)G\big) &= E\big((\partial_\gamma \varphi)(F) H_\beta(F,G)\big) \\ &= E\big(\varphi(F) H_\gamma(F, H_\beta(F,G))\big) \\ &= E\big(\varphi(F) H_\alpha(F,G)\big). \end{aligned}$$

The interest of this definition in connection with the study of probability laws can be deduced from the next result.

Proposition 1.1

1) *Assume that (1.1) holds for* $\alpha = (1, \dots, 1)$ *and* $G = 1$. *Then the probability law of* F *has a density* $p(x)$ *with respect to Lebesgue measure on* $\mathbb{R}^n$. *Moreover,*

$$p(x) = E\big(\mathbf{1}_{(x \le F)} H_{(1,\dots,1)}(F, 1)\big). \tag{1.2}$$

In particular, p *is continuous.*

2) *Assume that for any multiindex* α *the formula (1.1) holds true with* $G = 1$. *Then* $p \in \mathcal{C}^{|\alpha|}(\mathbb{R}^n)$ *and*

$$\partial_\alpha p(x) = (-1)^{|\alpha|} E\big(\mathbf{1}_{(x \le F)} H_{\alpha+1}(F, 1)\big), \tag{1.3}$$

where $\alpha + 1 := (\alpha_1 + 1, \dots, \alpha_d + 1)$.

Proof

We start by giving a non-rigorous argument which leads to the conclusion of part 1. Heuristically $p(x) = E\big(\delta_0(F - x)\big)$, where δ_0 denotes the *Dirac delta function*. The primitive of this distribution on $\mathbb{R}^n$ is $\mathbf{1}_{[0,\infty)}$. Thus by (1.1) we have

$$\begin{aligned} p(x) = E\big(\delta_0(F - x)\big) &= E\big((\partial_{1,\dots,1} \mathbf{1}_{[0,\infty)})(F - x)\big) \\ &= E\big(\mathbf{1}_{[0,\infty)}(F - x) H_{(1,\dots,1)}(F, 1)\big). \end{aligned}$$

Let us be more precise. Fix $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and set $\varphi(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f(y)\, dy$. Fubini's theorem yields

$$\begin{aligned} E\big(f(F)\big) = E\big((\partial_{1,\dots,1}\varphi)(F)\big) &= E\big(\varphi(F) H_{(1,\dots,1)}(F,1)\big) \\ &= E\left(\left(\int_{\mathbb{R}^n} \mathbf{1}_{(x \le F)} f(x)\, dx\right) H_{(1,\dots,1)}(F,1)\right) \\ &= \int_{\mathbb{R}^n} f(x) E\big(\mathbf{1}_{(x \le F)} H_{(1,\dots,1)}(F,1)\big). \end{aligned}$$

Let B be a bounded Borel set of $\mathbb{R}^n$. Consider a sequence of functions $f_n \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ converging pointwise to $\mathbf{1}_B$. Owing to the previous identities (applied to f_n) and Lebesgue bounded convergence we obtain

$$E\big(\mathbf{1}_B(F)\big) = \int_{\mathbb{R}^n} \mathbf{1}_B(x) E\big(\mathbf{1}_{(x\le F)} H_{(1,\dots,1)}(F,1)\big). \tag{1.4}$$

Hence the law of F is *absolutely continuous* and its density is given by (1.2). Since $H_{(1,\dots,1)}(F,1)$ is assumed to be in $L^1(\Omega)$, formula (1.2) implies the continuity of p, by bounded convergence. This finishes the proof of part 1.

The proof of part 2 is done recursively. For the sake of simplicity, we shall only give the details of the first iteration for the multiindex $\alpha = (1,\dots,1)$.

Let $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$,

$$\Phi(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f(y)\,dy, \qquad \Psi(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \Phi(y)\,dy.$$

By assumption,

$$\begin{aligned} E\big(f(F)\big) &= E\big(\Phi(F) H_{(1,\dots,1)}(F,1)\big) \\ &= E\Big(\Psi(F) H_{(1,\dots,1)}\big(F, H_{(1,\dots,1)}(F,1)\big)\Big) \\ &= E\big(\Psi(F) H_{(2,\dots,2)}(F,1)\big). \end{aligned}$$

Fubini's Theorem yields

$$\begin{aligned} &E\big(\Psi(F) H_{(2,\dots,2)}(F,1)\big) \\ &\quad = E\left(\int_{-\infty}^{F_1} dy_1 \cdots \int_{-\infty}^{F_n} dy_n \left(\int_{-\infty}^{y_1} dz_1 \cdots \int_{-\infty}^{y_n} dz_n f(z)\right) H_{(2,\dots,2)}(F,1)\right) \\ &\quad = E\left(\int_{-\infty}^{F_1} dz_1 \cdots \int_{-\infty}^{F_n} dz_n f(z) \int_{z_1}^{F_1} dy_1 \cdots \int_{z_n}^{F_n} dy_n H_{(2,\dots,2)}(F,1)\right) \\ &\quad = \int_{\mathbb{R}^n} dz f(z) E\left(\prod_{i=1}^n (F_i - z_i)^+ H_{(2,\dots,2)}(F,1)\right). \end{aligned}$$

This shows that the density of F is given by

$$p(x) = E\left(\prod_{i=1}^n (F_i - x_i)^+ H_{(2,\dots,2)}(F,1)\right),$$

by a limit argument, as in the first part of the proof. The function $x \mapsto \prod_{i=1}^n (F^i - x_i)^+$ is differentiable, except when $x_i = F_i$, almost surely. Therefore by bounded convergence

$$\partial_{(1,\dots,1)} p(x) = (-1)^n E\big(\mathbf{1}_{[x,\infty)}(F) H_{(2,\dots,2)}(F,1)\big). \qquad \square$$

Remark 1.1 The conclusion in part 2 of the preceding Proposition is quite easy to understand by formal arguments. Indeed, roughly speaking φ should be such that its derivative ∂_α is the Dirac delta function δ_0. Since taking primitives makes functions smoother, the higher $|\alpha|$ is, the smoother φ must be. Thus, having (1.1) for any multiindex α yields infinite differentiability for $p(x) = E\big(\delta_0(F - x)\big)$.

Malliavin, in the development of his theory, used the criteria given in the next Proposition for the existence and smoothness of density (see ref. [33]).

Proposition 1.2

1) *Assume that for any $i \in \{1, 2, \dots, n\}$ and every function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, there exist positive constants C_i, not depending on φ, such that*

$$\Big|E\big((\partial_i \varphi)(F)\big)\Big| \le C_i \|\varphi\|_\infty. \tag{1.5}$$

Then the law of F has a density.

2) *Assume that for any multiindex α and every function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ there exist positive constants C_α, not depending on φ, such that*

$$\Big|E\big((\partial_\alpha \varphi)(F)\big)\Big| \le C_\alpha \|\varphi\|_\infty. \tag{1.6}$$

Then the law of F has a $\mathcal{C}^\infty$ density.

Remark 1.2 Checking (1.5), (1.6) means that we have to eliminate the derivatives ∂_i, ∂_α and thus one is naturally led to an integration by parts procedure.

Remark 1.3 Malliavin formulates Proposition 1.2 in a more general setting. Indeed, instead of considering probability laws $P \circ F^{-1}$, he deals with finite measures μ on $\mathbb{R}^n$. The reader interested in the proof of this result is referred to references [33] and [43].

Comments

Comparing Propositions 1.1 and 1.2 leads to some comments:

1) Let $n = 1$. The assumption in part 1) of Proposition 1.1 implies (1.5). However, for $n > 1$, both hypotheses are not comparable. The conclusion in the former Proposition gives more information about the density than in the later one.

2) Let $n > 1$. Assume that (1.1) holds for any multiindex α with $|\alpha| = 1$. Then, by the recursivity of the integration by parts formula, we obtain the validity of (1.1) for $\alpha = (1, \dots, 1)$.

3) Since the random variable $H_\alpha(F, G)$ in (1.1) is assumed to belong to $L^1(\Omega)$, the identity (1.1) with $G = 1$ clearly implies (1.6). Therefore the assumption in part 2 of Proposition 1.1 is stronger than in Proposition 1.2. But, on the other hand, the conclusion is more precise.

Chapter 2

Finite Dimensional Malliavin Calculus

In this chapter, we shall consider random vectors defined on the probability space $\big(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m), \mu_m\big)$, where μ_m is the *standard Gaussian measure*, that is

$$\mu_m(dx) = (2\pi)^{-\frac{m}{2}} \exp\left(-\frac{|x|^2}{2}\right) dx.$$

We denote by E_m the expectation with respect to the measure μ_m. Consider a random vector $F : \mathbb{R}^m \to \mathbb{R}^n$. The purpose is to find sufficient conditions ensuring absolute continuity with respect to the Lebesgue measure on $\mathbb{R}^n$ of the probability law of F and the smoothness of the density. More precisely, we would like to obtain expressions such as (1.1). This will be done in a quite sophisticated way as a prelude to the methodology to be applied in the infinite dimensional case. For the sake of simplicity, we shall only deal with multiindices α of order one. That means that we shall only address the problem of existence of density for the random vector F.

2.1 The Ornstein-Uhlenbeck operator

Let $(B_t, t \geq 0)$ be a standard $\mathbb{R}^m$-valued *Brownian motion*. Consider the *linear stochastic differential equation*

$$dX_t(x) = \sqrt{2}\, dB_t - X_t(x)\, dt, \tag{2.1}$$

with initial condition $x \in \mathbb{R}^m$. Using the *Itô formula*, it is immediate to check that the solution to (2.1) is given by

$$X_t(x) = \exp(-t)x + \sqrt{2} \int_0^t \exp\big(-(t-s)\big)\, dB_s. \tag{2.2}$$

The operator *semigroup* associated with the Markov process solution to (2.1) is defined by $P_t f(x) = E_m f\big(X_t(x)\big)$. Notice that the law of $Z_t(x) = \sqrt{2}\int_0^t \exp\big(-(t-s)\big)\, dB_s$ is Gaussian, mean zero and with covariance given by $\big(1-\exp(-2t)\big)I$. This fact, together with (2.2), yields

$$P_t f(x) = \int_{\mathbb{R}^m} f\left(\exp(-t)x + \sqrt{1-\exp(-2t)}\,y\right) \mu_m(dy). \tag{2.3}$$

We are going to identify the class of functions f for which the right hand-side of (2.3) makes sense ; and we will also compute the infinitesimal generator of the semigroup.

Lemma 2.1

We have the following facts about the semigroup which is generated by $(X_t, t \geq 0)$:

1) $(P_t, t \geq 0)$ *is a* ***contraction semigroup*** *on* $L^p(\mathbb{R}^m; \mu_m)$*, for all* $p \geq 1$.

2) *For any* $f \in \mathcal{C}_b^2(\mathbb{R}^m)$ *and every* $x \in \mathbb{R}^m$,

$$\lim_{t\to 0} \frac{1}{t}\big(P_t f(x) - f(x)\big) = L_m f(x), \tag{2.4}$$

where $L_m = \triangle - x\cdot\nabla = \sum_{i=1}^m \partial^2_{x_i x_i} - \sum_{i=1}^m x_i \partial_{x_i}$.

3) $(P_t, t \geq 0)$ *is a* ***symmetric semigroup*** *on* $L^2(\mathbb{R}^m; \mu_m)$.

Proof

1) Let X and Y be independent random variables with law μ_m. The law of $\exp(-t)X + \sqrt{1-\exp(-2t)}Y$ is also μ_m. Therefore, $(\mu_m \times \mu_m)\circ T^{-1} = \mu_m$, where $T(x,y) = \exp(-t)x + \sqrt{1-\exp(-2t)}y$. Then, the definition of $P_t f$ and this remark yields

$$\begin{aligned}\int_{\mathbb{R}^m} \left|P_t f(x)\right|^p \mu_m(dx) &\leq \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \left|f\big(T(x,y)\big)\right|^p \mu_m(dx)\mu_m(dy)\\ &= \int_{\mathbb{R}^m} \left|f(x)\right|^p \mu_m(dx).\end{aligned}$$

2) This follows very easily by applying the Itô formula to the process $f(X_t)$.

3) We must prove that for any $g \in L^2(\mathbb{R}^m; \mu_m)$,

$$\int_{\mathbb{R}^m} P_t f(x) g(x) \mu_m(dx) = \int_{\mathbb{R}^m} f(x) P_t g(x) \mu_m(dx),$$

or equivalently

$$E_m\left(f\left(\exp(-t)X + \sqrt{1-\exp(-2t)}Y\right) g(X)\right)$$
$$= E_m\left(g\left(\exp(-t)X + \sqrt{1-\exp(-2t)}Y\right) f(X)\right),$$

where X and Y are two independent standard Gaussian variables. This follows easily from the fact that the vector (Z, X), where

$$Z = \exp(-t)X + \sqrt{1-\exp(-2t)}Y,$$

has a Gaussian distribution and each component has law μ_m. □

The appropriate spaces to perform the integration by parts mentioned above are defined in terms of the *eigenvalues* of the operator L_m. We are going to compute these eigenvalues using the Hermite polynomials. In the next chapter, we shall exploit this relationship in a stochastic framework. The *Hermite polynomials* $H_n(x)$, $x \in \mathbb{R}$, $n \geq 0$ are defined as follows:

$$\exp\left(-\frac{t^2}{2} + tx\right) = \sum_{n=0}^{\infty} t^n H_n(x). \tag{2.5}$$

That is,

$$H_n(x) = \frac{1}{n!}\frac{d^n}{dt^n} \exp\left(-\frac{t^2}{2} + tx\right)\bigg|_{t=0}$$
$$= \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \exp\left(-\frac{x^2}{2}\right). \tag{2.6}$$

Notice that $H_0(x) = 1$ and $H_n(x)$ is a polynomial of degree n, for any $n \geq 1$. Hence, any polynomial can be written as a sum of Hermite polynomials and therefore the set $(H_n, n \geq 0)$ is dense in $L^2(\mathbb{R}, \mu_1)$.

Moreover,

$$E_1(H_n H_m) = \frac{1}{(n!m!)^{\frac{1}{2}}} \delta_{n,m},$$

where $\delta_{n,m}$ denotes the *Kronecker symbol*. Indeed, this is a consequence of the identity

$$E_1\left(\exp\left(sX - \frac{s^2}{2}\right) \exp\left(tX - \frac{t^2}{2}\right)\right) = \exp(st),$$

which is proved by a direct computation. Thus, $\left(\sqrt{n!}H_n, n \geq 0\right)$ is a complete *orthonormal system* of $L^2(\mathbb{R}, \mu_1)$.

One can easily check that

$$H_n'(x) = H_{n-1}(x),$$
$$(n+1)H_{n+1}(x) = xH_n(x) - H_n'(x).$$

Thus,

$$L_1 H_n(x) := H_n''(x) - xH_n'(x) = -nH_n(x).$$

Therefore, the operator L_1 is non positive, $(H_n, n \geq 0)$ is the sequence of *eigenfunctions* and $(-n, n \geq 0)$ the corresponding sequence of eigenvalues. The generalisation to any finite dimension $m \geq 1$ is not difficult. Indeed, let $a = (a_1, a_2, \dots,)$, $a_i \in \mathbb{N}$, be a multiindex. Assume that $a_i = 0$ for any $i > m$. We define the *generalized Hermite polynomial* $H_a(x)$, $x \in \mathbb{R}^m$, by

$$H_a(x) = \prod_{i=1}^{\infty} H_{a_i}(x_i).$$

Set $|a| = \sum_{i=1}^m a_i$ and define $L_m = \sum_{i=1}^m L_1^i$, with $L_1^i = \partial^2_{x_i x_i} - x_i \partial_{x_i}$. Then

$$L_m\big(H_a(x)\big) = \sum_{i=1}^m \left(\prod_{j \neq i} H_{a_j}(x_j)(-a_i) H_{a_i}(x_i) \right) = -|a| H_a(x).$$

Therefore, the eigenvalues of L_m are again $(-n, n \geq 0)$ and the corresponding sequence of *eigenspaces* are those generated by the sets

$$\left(\prod_{i=1}^m \sqrt{a_i!} H_{\alpha_i}(x_i),\ \sum_{i=1}^m \alpha_i = n,\ \alpha_i \geq 0 \right).$$

Notice that if $|a| = n$, then $H_a(x)$ is a polynomial of degree n. Denote by $\mathcal{P}_m$ the set of polynomials on $\mathbb{R}^m$. Fix $p \in [1, \infty)$ and $k \geq 0$. We define a seminorm on $\mathcal{P}_m$ as follows:

$$\|F\|_{k,p} = \left\| (I - L_m)^{\frac{k}{2}} F \right\|_{L^p(\mu_m)}, \tag{2.7}$$

where for any $s \in \mathbb{R}$, the operator $(I - L_m)^s$ is defined using the *spectral decomposition* of L_m.

Lemma 2.2

1) Let $k \leq k'$, $p \leq p'$, $k, k' \geq 0$, $p, p' \in [1, \infty)$. Then for any $F \in \mathcal{P}_m$,

$$\|F\|_{k,p} \leq \|F\|_{k',p'}. \tag{2.8}$$

2) The norms $\|\cdot\|_{k,p}$, $k \geq 0$, $p \in [1,\infty)$, are compatible in the following sense: If $(F_n, n \geq 1)$ is a sequence in $\mathcal{P}_m$ such that $\lim_{n\to 0} \|F_n\|_{k,p} = 0$ and it is a Cauchy sequence in the norm $\|\cdot\|_{k',p'}$, then $\lim_{n\to 0} \|F_n\|_{k',p'} = 0$.

Proof

1) Clearly, by Hölder's inequality the statement holds true for $k = k'$. Hence it suffices to check that $\|F\|_{k,p} \leq \|F\|_{k',p}$, for any $k \leq k'$. To this end we prove that for any $\alpha \geq 0$,

$$\left\|(I - L_m)^{-\alpha} F\right\|_{L^p(\mu_m)} \leq \|F\|_{L^p(\mu_m)}. \tag{2.9}$$

Fix $F \in \mathcal{P}_m$. Consider its decomposition in $L^2(\mathbb{R}^m;\mu_m)$ with respect to the orthonormal basis given by the Hermite polynomials, $F = \sum_{n=0}^{\infty} J_n F$. Since L_m is the *infinitesimal generator* of P_t, the formal relationship $P_t = \exp(L_m)$ yields $P_t F = \sum_{n=0}^{\infty} \exp(-nt) J_n F$.

The obvious identity

$$(1+n)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty \exp\big(-t(n+1)\big) t^{\alpha-1}\, dt,$$

valid for any $\alpha > 0$, yields

$$\begin{aligned}
(I - L_m)^{-\alpha} F &= \sum_{n=0}^{\infty} (1+n)^{-\alpha} J_n F \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty \exp(-t) t^{\alpha-1} \sum_{n=0}^{\infty} \exp(-nt) J_n F\, dt \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty \exp(-t) t^{\alpha-1} P_t F\, dt.
\end{aligned}$$

Hence, the contraction property of the semigroup P_t yields

$$\begin{aligned}
\left\|(I - L_m)^{-\alpha} F\right\|_{L^p(\mu_m)} &\leq \frac{1}{\Gamma(\alpha)} \int_0^\infty \exp(-t) t^{\alpha-1} \|P_t F\|_{L^p(\mu_m)} \\
&\leq \|F\|_{L^p(\mu_m)}.
\end{aligned}$$

Fix $0 \leq k \leq k'$. Using (2.9) we obtain

$$\begin{aligned}
\left\|(I - L_m)^{\frac{k}{2}} F\right\|_{L^p(\mu_m)} &= \left\|(I - L_m)^{\frac{k-k'}{2}} (I - L_m)^{\frac{k'}{2}} F\right\|_{L^p(\mu_m)} \\
&\leq \left\|(I - L_m)^{\frac{k'}{2}} F\right\|_{L^p(\mu_m)}.
\end{aligned}$$

2) Set $G_n = (I-L)^{\frac{k'}{2}} F_n \in \mathcal{P}_m$. By assumption, $(G_n, n \geq 1)$ is a Cauchy sequence in $L^{p'}(\mu_m)$. Let us denote by G its limit. We want to check that $G = 0$. Let $H \in \mathcal{P}_m$, then

$$\begin{aligned}\int_{\mathbb{R}^m} GH d\mu_m &= \lim_{n\to\infty} \int_{\mathbb{R}^m} G_n H \, d\mu_m \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^m} (I-L)^{\frac{k-k'}{2}} G_n (I-L)^{\frac{k'-k}{2}} H \, d\mu_m = 0.\end{aligned}$$

Since $\mathcal{P}_m$ is dense in $L^q(\mu_m)$, for any $q \in [1, \infty)$ (see for instance ref. [21]), we conclude that $G = 0$. This ends the proof of the Lemma. □

Let $\mathbb{D}_m^{k,p}$ be the completion of the set $\mathcal{P}_m$ with respect to the norm $\| \cdot \|_{k,p}$ defined in (2.7). Set

$$\mathbb{D}_m^\infty = \bigcap_{p\geq 1} \bigcap_{k \geq 0} \mathbb{D}_m^{k,p}.$$

Lemma 2.2 ensures that the set $\mathbb{D}_m^\infty$ is well defined. Moreover, it is easy to check that $\mathbb{D}_m^\infty$ is an algebra.

REMARK 2.1 Let $F \in \mathbb{D}_m^\infty$. Consider a sequence $(F_n, n \geq 1) \subset \mathcal{P}_m$ converging to F in the topology of $\mathbb{D}_m^\infty$, that is

$$\lim_{n\to\infty} \big\| F - F_n \big\|_{k,p} = 0,$$

for any $k \geq 0$, $p \in [1, \infty)$. Then $L_m F$ is defined as the limit in the topology of $\mathbb{D}_m^\infty$ of the sequence $F - (I - L_m) F_n$.

2.2 The adjoint of the differential

We are looking for an operator δ_m which can be considered as the adjoint of the gradient ∇ in $L^2(\mathbb{R}^m, \mu_m)$. Such an operator must act on functions $\varphi : \mathbb{R}^m \to \mathbb{R}^m$, take values in the space of real-valued functions defined on $\mathbb{R}^m$ and satisfy the duality relation

$$E_m \langle \nabla f, \varphi \rangle = E_m (f \delta_m \varphi), \tag{2.10}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^m$. Assume first that $f, \varphi^i \in \mathcal{P}_m$, $i = 1, \dots, m$. Then, an *integration by parts* yields

$$\begin{aligned}E_m \langle \nabla f, \varphi \rangle &= \sum_{i=1}^m \int_{\mathbb{R}^m} \partial_i f(x) \varphi^i(x) \mu_m(dx) \\ &= \sum_{i=1}^m \int_{\mathbb{R}^m} f(x) \big(x_i \varphi^i(x) - \partial_i \varphi^i(x) \big) \mu_m(dx).\end{aligned}$$

Hence

$$\delta_m \varphi = \sum_{i=1}^{m} (x_i \varphi^i - \partial_i \varphi^i). \tag{2.11}$$

Notice that $\delta_m \circ \nabla = -L_m$.

The definition (2.11) yields the next useful formula

$$\delta_m(f \nabla g) = -\langle \nabla f, \nabla g \rangle - f L_m g, \tag{2.12}$$

for any $f, g \in \mathcal{P}_m$.

We remark that the operator δ_1 satisfies

$$\begin{aligned} \delta_1 H_n(x) &= x H_n(x) - H_n'(x) = x H_n(x) - H_{n-1}(x) \\ &= (n+1) H_{n+1}(x). \end{aligned}$$

Therefore it increases the order of a Hermite polynomial by one.

REMARK 2.2 All the above identities make sense for $f, g \in \mathbb{D}_m^\infty$. Indeed, it suffices to justify that one can extend the operator ∇ to $\mathbb{D}_m^\infty$. Here is one possible argument:

Let $\mathcal{S}(\mathbb{R}^m)$ be the set of Schwartz test functions. Consider the isometry $J : L^2(\mathbb{R}^m, \lambda_m) \to L^2(\mathbb{R}^m, \mu_m)$ defined by

$$J(f)(x) = f(x)(2\pi)^{\frac{m}{4}} \exp\left(\frac{|x|^2}{4}\right),$$

where λ_m denotes Lebesgue measure on $\mathbb{R}^m$. Following reference [56] (page 142), $\bigcap_{k \geq 0} \mathbb{D}_m^{2,k} = J\big(\mathcal{S}(\mathbb{R}^m)\big)$. Then, for any $F \in \mathbb{D}_m^\infty$ there exists $\tilde{F} \in \mathcal{S}(\mathbb{R}^m)$ such that $F = J(\tilde{F})$ and one can define $\nabla F = J(\nabla \tilde{F})$.

We will see in the next chapter that Meyer's result on *equivalence of norms* shows that the infinite-dimensional analogue of the spaces $\mathbb{D}_m^{k,p}$ are the suitable spaces where the Malliavin k-th derivative makes sense.

2.3 An integration by parts formula: Existence of a density

Let $F : \mathbb{R}^m \to \mathbb{R}^n$ be a random vector, $F = (F^1, \ldots, F^n)$. We assume that $F \in \mathbb{D}_m^\infty(\mathbb{R}^n)$; that is, $F^i \in \mathbb{D}_m^\infty$, for any $i = 1, \ldots, n$. The ***Malliavin matrix*** of F — also called *covariance matrix* — is defined by

$$A(x) = \big(\langle \nabla F^i(x), \nabla F^j(x) \rangle\big)_{1 \leq i,j \leq n}.$$

Notice that by its very definition, $A(x)$ is a symmetric, non-negative definite matrix, for any $x \in \mathbb{R}^m$. Clearly $A(x) = DF(x)DF(x)^T$, where $DF(x)$ is the Jacobian matrix at x and the superscript T means the transpose.

We want to give sufficient conditions ensuring existence of a density for $P \circ F^{-1}$. We shall apply the criterium of part 1) of Proposition 1.2.

Let us perform some computations showing that $(\partial_i \varphi)(F)$, $i = 1, \dots, n$, satisfies a linear system of equations. Indeed, by the chain rule,

$$\begin{aligned}\left\langle \nabla\Big(\varphi\big(F(x)\big)\Big), \nabla F^l(x) \right\rangle &= \sum_{j=1}^{m}\sum_{k=1}^{n} (\partial_k \varphi)\big(F(x)\big)\partial_j F^k(x)\partial_j F^l(x) \\ &= \sum_{k=1}^{n} \langle \nabla F^l(x), \nabla F^k(x)\rangle (\partial_k\varphi)\big(F(x)\big) \\ &= \Big(A(x)(\nabla^T\varphi)\big(F(x)\big)\Big)_l, \end{aligned} \tag{2.13}$$

$l = 1, \dots, n$. Assume that the matrix $A(x)$ is inversible μ_m-almost everywhere. Then one gets

$$(\partial_i\varphi)(F) = \sum_{l=1}^{n} \left\langle \nabla\Big(\varphi\big(F(x)\big)\Big), A^{-1}_{i,l}(x)\nabla F^l(x) \right\rangle, \tag{2.14}$$

for every $i = 1, \dots, n$, μ_m-almost everywhere.

Taking expectations formally and using (2.12), (2.14) yields

$$\begin{aligned} E_m\big((\partial_i\varphi)(F)\big) &= \sum_{l=1}^{n} E_m\left\langle \nabla\big(\varphi(F)\big), A^{-1}_{i,l}\nabla F^l \right\rangle \\ &= \sum_{l=1}^{n} E_m\Big(\varphi(F)\delta_m\big(A^{-1}_{i,l}\nabla F^l\big)\Big) \\ &= \sum_{l=1}^{n} E_m\Big(\varphi(F)\Big(-\langle \nabla A^{-1}_{i,l}, \nabla F^l\rangle - A^{-1}_{i,l} L_m F^l\Big)\Big). \end{aligned} \tag{2.15}$$

Formula (2.15) shows that

$$E_m\big(\partial_i\varphi(F)\big) = E_m\big(\varphi(F)H_i(F,1)\big), \tag{2.16}$$

with

$$H_i(F,1) = \sum_{l=1}^{n} \delta_m\big(A^{-1}_{i.l}\nabla F^l\big)$$
$$= -\sum_{l=1}^{n}\Big(\big\langle \nabla A^{-1}_{i,l}, \nabla F^l\big\rangle + A^{-1}_{i,l} L_m F^l\Big). \tag{2.17}$$

This is an integration by parts formula as in Definition 1.1.

For higher differential orders, things are a little bit more difficult, but essentialy the same ideas would lead to the analogue of formula (1.1) with $\alpha = (1,\dots,1)$ and $G = 1$.

The preceding discussion and Proposition 1.2 yield the following result.

Proposition 2.1

Let $F \in \mathbb{D}^\infty_m(\mathbb{R}^n)$. Assume that:

1) The matrix $A(x)$ is invertible for every $x \in \mathbb{R}^m$, μ_m-almost everywhere.

2) $\det A^{-1} \in L^p(\mathbb{R}^m;\mu_m)$, $\nabla(\det A^{-1}) \in L^r(\mathbb{R}^m;\mu_m)$, for some $p, r \in (1,\infty)$.

Then the law of F is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$.

Proof

The assumptions in 2) show that

$$C_i := \sum_{l=1}^{n} E_m\left(\Big|\big\langle \nabla A^{-1}_{i,l}, \nabla F^l\big\rangle\Big| + \big|A^{-1}_{i,l} L_m F^l\big|\right)$$

is finite. Therefore, one can take expectations on both sides of (2.14). By (2.15), it follows that

$$\big|E_m(\delta_i\varphi)(F)\big| \le C_i \|\varphi\|_\infty.$$

This finishes the proof of the Proposition. □

Remark 2.3 The proof of smoothness properties for the density requires an iteration of the procedure presented in the proof of the Proposition 2.1.

Comments

Developing Malliavin Calculus on a finite dimensional Gaussian space is a stimulating exercise which gives a preliminary and useful insight into this very intricate topic. Is also provides good training material, since computations can be carried out explicitely. Stroock's course (ref. [63]) insists on the finite dimensional setting before entering into the core of the Calculus; Ocone follows the same strategy in reference [48]. We have followed essentially his presentation. The proof of Lemma 2.2 can be found in reference [68] in the general infinite dimensional framework.

2.4 Exercises

2.4.1

Let $f, g \in \mathbb{D}_m^\infty$ and define

$$\Gamma(f,g) = L_m(fg) - fL_m g - gL_m f.$$

Show that

1) $\Gamma(f,g) = 2\langle \nabla f, \nabla g\rangle$.

2) $E_m\big(\Gamma(f,g)\big) = -E_m(fL_m g)$.

Identify the operator $\delta_m \circ \nabla$.

Hint: Apply the identity (2.12).

2.4.2

Prove that $P_t(H_n) = \exp(-nt)H_n$.

Hint: Using the definition of P_t and the *Laplace transform* of the Gaussian measure, check that

$$P_t\left(\exp\left(-\frac{t^2}{2} + tx\right)\right) = \exp\left(-\frac{\left(te^{-t}\right)^2}{2} + te^{-t}x\right).$$

Chapter 3

The Basic Operators of Malliavin Calculus

In this chapter we introduce the three basic operators needed to develop the infinite dimensional Malliavin calculus on a Gaussian space: The *Malliavin derivative*, its adjoint — the *divergence operator* — and the *Ornstein-Uhlenbeck operator*.

We start by describing the underlying probability space. Let H be a real separable Hilbert space. Denote by $\|\cdot\|_H$ and $\langle \cdot\,,\,\cdot\rangle_H$ the norm and the inner product, respectively. There exist a probability space $(\Omega, \mathcal{G}, \mu)$ and a family $\mathcal{M} = \big(W(h), h \in H\big)$ of random variables defined on this space, such that the mapping $h \to W(h)$ is linear, each $W(h)$ is Gaussian, $EW(h) = 0$ and $E\big(W(h_1)W(h_2)\big) = \langle h_1, h_2\rangle_H$ (see for instance Proposition 1.3 of Chap. 1 of ref. [57]). Such a family is constructed as follows. Let $(e_n, n \geq 1)$ be a *complete orthonormal system* in H. Consider the *canonical probability space* $(\Omega, \mathcal{G}, P)$ associated with a sequence $(g_n, n \geq 1)$ of standard independent Gaussian random variables. That is, $\Omega = \mathbb{R}^{\otimes \mathbb{N}}$, $\mathcal{G} = \mathcal{B}^{\otimes \mathbb{N}}$, $\mu = \mu_1^{\otimes \mathbb{N}}$ where, according to the notations of Chapter 1, μ_1 denotes the standard Gaussian measure on $\mathbb{R}$. For each $h \in H$, the series $\sum_{n\geq 1} \langle h, e_n\rangle_H g_n$ converges in $L^2(\Omega, \mathcal{G}, \mu)$ to a random variable that we denote by $W(h)$. Notice that the set $\mathcal{M}$ is a closed *Gaussian subspace* of $L^2(\Omega)$ that is isometric to H. In the sequel, we shall assume that $\mathcal{G}$ is the σ-field generated by $\mathcal{M}$.

Here is an example of such a *Gaussian family*. Let $H = L^2(A, \mathcal{A}, m)$, where $(A, \mathcal{A}, m)$ is a separable σ-finite, atomless measure space. For any $F \in \mathcal{A}$ with $m(F) < \infty$, set $W(F) = W(\mathbf{1}_F)$. The stochastic Gaussian process $\big(W(F), F \in \mathcal{A}, m(F) < \infty\big)$ is such that $W(F)$ and $W(G)$ are independent if F and G are disjoint sets; in this case, $W(F \cup G) = W(F) + W(G)$. Following reference [67], we call such a process a ***white noise*** *based on* m. Then the random variable $W(h)$ coincides with the first order *Itô stochastic integral* $\int_A h(t)W(dt)$ with respect to W (see ref. [20]). For instance, if $A = \mathbb{R}_+$, $\mathcal{A}$ is the σ-field of Borel sets of $\mathbb{R}_+$ and m is the Lebesgue measure on $\mathbb{R}_+$, then $W(h) = \int_0^\infty h(t)\,dW_t$ —the Itô integral of a deterministic integrand— where $(W_t, t \geq 0)$ is a standard Brownian motion. In Chapter 5 we shall introduce another important class of Gaussian families indexed by two parameters representing time and space, respectively; the time covariance is given by the Lebesgue measure while the space correlation is homogeneous and is given by some kind of measure. We will refer to these processes as *noises that are white in time and spatially correlated*.

3.1 The Ornstein-Uhlenbeck operator

We could introduce this operator following exactly the same approach as that used in Section 2.1 for the finite dimensional case. However, we shall avoid introducing infinite dimensional evolution equations. For this reason, we shall start with the analogue of the formula (2.3) which, in this context, is called *Mehler's formula*. For any $F \in L^p(\Omega;\mu)$, $p \geq 1$, set

$$P_t F(\omega) = \int_\Omega F\Big(\exp(-t)\omega + \sqrt{1-\exp(-2t)}\,\omega'\Big)\,\mu(d\omega'), \tag{3.1}$$

$t \geq 0$. We have the following.

Proposition 3.1

The above formula (3.1) defines a positive symmetric contraction semigroup on $L^p(\Omega;\mu)$ and satisfies $P_t 1 = 1$.

PROOF

The contraction property and the symmetry are proved following exactly the same arguments as in finite dimension, replacing the measure μ_m by μ (see Lemma 2.1). Positivity is obvious, as well as the property $P_t 1 = 1$.

Let us now prove the semigroup property. Let $s, t \geq 0$. Then

$$
\begin{aligned}
&P_t(P_sF)(\omega) \\
&= \int_\Omega \mu(d\omega')(P_sF)\Big(e^{-t}\omega + \sqrt{1-e^{-2t}}\,\omega'\Big) \\
&= \int_\Omega \int_\Omega \mu(d\omega')\mu(d\omega'')F\Big(e^{-(s+t)}\omega + e^{-s}\sqrt{1-e^{-2t}}\,\omega' + \sqrt{1-e^{-2s}}\,\omega''\Big) \\
&\qquad \times \int_\Omega \mu(d\omega'')F\Big(e^{-(s+t)}\omega + \sqrt{1-e^{-2(t+s)}}\,\omega''\Big) \\
&= P_{t+s}F(\omega).
\end{aligned}
$$

This finishes the proof of the proposition. □

REMARK 3.1 The operator semigroup $\{P_t, t \geq 0\}$ satisfies a stronger property than contraction. Indeed, Nelson [42] has proved that if $q(t) = e^{2t}(p-1)+1$, $t > 0$, $p \geq 1$, then

$$\|P_tF\|_{q(t)} \leq \|F\|_p,$$

where for any $q \geq 1$, the notation $\|\cdot\|_q$ denotes the $L^q(\Omega, \mathcal{G}, \mu)$-norm. Notice that $q(t) > p$. This property is called ***hypercontractivity***.

In order to describe the infinitesimal operator L of the semigroup (P_t) in an operational way, we give its spectral decomposition. To this end we shall introduce the Wiener chaos decomposition of a random variable in $L^2(\Omega, \mathcal{G}, \mu)$. This is the infinite-dimensional analogue of the decomposition of a function in $L^2(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m), \mu_m)$ in the basis consisting of the Hermite polynomials.

Fix a multiindex $a = (a_1, a_2, \ldots)$, $a_i \in \mathbb{Z}_+$, $a_i = 0$ except for a finite number of indices i. Set $|a| = \sum_i |a_i|$. We define the random variable

$$H_a = \sqrt{a!}\prod_{i=1}^\infty H_{a_i}\big(W(e_i)\big), \tag{3.2}$$

where $a! = \prod_{i=1}^\infty a_i!$ and H_{a_i} is the Hermite polynomial defined in (2.5).

Let $\mathcal{P}$ be the class of random variables of the form

$$F = f\big(W(h_1), \ldots, W(h_n)\big), \tag{3.3}$$

$n \geq 1$, where f is a polynomial function. It is easy to check that $\mathcal{P}$ is dense in $L^2(\Omega, \mathcal{G}, \mu)$.

Lemma 3.1

The family (H_a) is an orthonormal basis of $L^2(\Omega, \mathcal{G}, \mu)$.

PROOF

By the definition of H_a and the independence of the random variables $W(e_i)$, one has

$$\int_\Omega H_a(\omega)H_b(\omega)\mu(d\omega) = \sqrt{a!}\sqrt{b!}\prod_{i=1}^{\infty}\int_\Omega H_{a_i}\big(W(e_i)\big)H_{b_i}\big(W(e_i)\big)\mu(d\omega)$$
$$= \sqrt{a!}\sqrt{b!}\prod_{i=1}^{\infty}\int_{\mathbb{R}} H_{a_i}(x)H_{b_i}(x)\mu_1(dx)$$
$$= \prod_{i=1}^{\infty}\delta_{a_i,b_i} = \delta_{a,b}.$$

Since $\mathcal{P}$ is dense in $L^2(\Omega, \mathcal{G}, \mu)$, this orthonormal family is total. □

Let $\mathcal{H}_n$ be the closed subspace of $L^2(\Omega, \mathcal{G}, \mu)$ generated by $\big(H_a, |a| = n\big)$. It is called the n-th *Wiener chaos*. By the previous lemma, the spaces $\mathcal{H}_n$ are orthogonal for different values of n. The following decomposition holds:

$$L^2(\Omega) = \bigoplus_{n=0}^{\infty}\mathcal{H}_n.$$

We will denote by J_n the *orthogonal projection* from $L^2(\Omega)$ onto $\mathcal{H}_n$.

REMARK 3.2 If $H = L^2(A, \mathcal{A}, m)$, then $J_n(F)$ can be written as a *multiple Itô integral* (see ref. [20]).

Proposition 3.2

Let $F \in L^2(\Omega, \mathcal{G}, \mu)$. Then

$$P_t(F) = \sum_{n=0}^{\infty} e^{-nt}J_n(F). \tag{3.4}$$

Proof

It suffices to prove (3.4) for random variables of the kind $F = \exp\big(\lambda W(h) - \frac{\lambda^2}{2}\big)$, where $h \in H$, $\|h\|_H = 1$, $\lambda \in \mathbb{R}$. Indeed, once the result is proved for such random variables we obtain $P_t H_n\big(W(h)\big) = e^{-nt} H_n\big(W(h)\big)$, for any $n \geq 0$, $h \in H$, and this suffices to identify the action of P_t on any Wiener chaos.

By the definition of $P_t F$ we have

$$\begin{aligned} P_t F &= \int_{\mathbb{R}} \exp\left(e^{-t} \lambda W(h) + \sqrt{1 - e^{-2t}}\, \lambda x - \frac{\lambda^2}{2} \right) \mu_1(dx) \\ &= \exp\left(e^{-t} \lambda W(h) - \frac{e^{-2t} \lambda^2}{2} \right). \end{aligned}$$

In terms of the Hermite polynomials, this last expresion is equal to

$$\sum_{n=0}^{\infty} e^{-nt} \lambda^n H_n\big(W(h)\big) = \sum_{n=0}^{\infty} e^{-nt} J_n F.$$

But $J_n(F) = \lambda^n H_n\big(W(h)\big)$, therefore the Proposition is proved. □

Definition 3.1 The *Ornstein-Uhlenbeck operator* L is the infinitesimal generator of the semigroup $(P_t, t \geq 0)$.

We are going to prove that

$$\operatorname{Dom} L = \left\{ F \in L^2(\Omega) : \sum_{n=1}^{\infty} n^2 \big\| J_n F \big\|_2^2 < \infty \right\},$$

where $\|\cdot\|_2$ denotes the $L^2(\Omega)$-norm, and

$$LF = \sum_{n=0}^{\infty} (-n) J_n(F). \tag{3.5}$$

Indeed, assume first that F satisfies the condition $\sum_{n=1}^{\infty} n^2 \big\| J_n F \big\|_2^2 < \infty$. Then the operator L defined by (3.5) makes sense and satisfies

$$E\left(\left| \frac{1}{t} (P_t F - F) - LF \right|^2 \right) = \sum_{n=0}^{\infty} \left(\frac{1}{t} (e^{-nt} - 1) + n \right)^2 \big\| J_n F \big\|_2^2.$$

This last expresion tends to zero as $t \to 0$. In fact, $\lim_{t \to 0} \frac{1}{t}(e^{-nt} - 1) + n = 0$ and $\frac{1}{t}(e^{-nt} - 1) \leq n$. Thus the result follows by bounded convergence. This shows that L is the infinitesimal generator of the semigroup $(P_t, t \geq 0)$. Conversely, assume that $\lim_{t\to 0} \frac{1}{t}(P_t F - F) = G$ in $L^2(\Omega)$. Then clearly,

$$J_n G = \lim_{t\to 0} \frac{1}{t} (P_t J_n F - J_n F) = -n J_n F.$$

Therefore, F satisfies $\sum_{n=1}^{\infty} n^2 \big\| J_n F\big\|_2^2 < \infty$ and $LF = G$.

For any $F \in \mathcal{P}$, $p \in [1,\infty)$, $k \in \mathbb{Z}_+$, consider the seminorm

$$\|F\|_{k,p} = \big\|(I-L)^{\frac{k}{2}} F\big\|_p. \tag{3.6}$$

Note that $(I-L)^{\frac{k}{2}} F = \sum_{n=0}^{\infty} (1+n)^{\frac{k}{2}} J_n F$. Definition (3.6) is the infinite dimensional analogue of (2.7). The results stated in Lemma 2.2 also hold in our setting. In fact the proofs are exactly the same with μ_m replaced by μ (see ref. [68]). Let $\mathbb{D}^{k,p}$ be the completion of the set $\mathcal{P}$ with respect to the norm $\|\cdot\|_{k,p}$ defined in (3.6). Set

$$\mathbb{D}^{\infty} = \bigcap_{p\geq 1} \bigcap_{k\geq 0} \mathbb{D}^{k,p}.$$

This set is an algebra. Notice that, as in the finite dimensional case (see Remark 2.1), we can extend the definition of the operator L to any random variable in $\mathbb{D}^{\infty}$.

3.2 The derivative operator

In this section, we introduce the infinite-dimensional version of the gradient operator. The idea shall be to start with *finite dimensional* random variables in a sense to be made precise and then, by a density argument, to extend the definition to a larger class of random variables. Let $\mathcal{S}$ be the set of random variables of the form

$$F = f\big(W(h_1), \dots, W(h_n)\big), \tag{3.7}$$

with $f \in \mathcal{C}_p^{\infty}(\mathbb{R}^n)$, $h_1, \dots, h_n \in H$, $n \geq 1$. Sometimes, we shall take $f \in \mathcal{C}_b^{\infty}(\mathbb{R}^n)$; in this case we shall write $\mathcal{S}_b$ instead of $\mathcal{S}$. The elements of $\mathcal{S}$ are called ***smooth functionals***.

We define the operator D on $\mathcal{S}$, with values in the set of H-valued random variables, by

$$DF = \sum_{i=1}^{n} \partial_i f\big(W(h_1), \dots, W(h_n)\big) h_i. \tag{3.8}$$

Fix $h \in H$ and set

$$F^{\epsilon h} = f\Big(W(h_1) + \epsilon\big\langle h, h_1\big\rangle_H, \dots, W(h_n) + \epsilon\big\langle h, h_n\big\rangle_H\Big),$$

$\epsilon > 0$. Then it is immediate to check that $\langle DF, h\rangle_H = (d/d\epsilon)F^{\epsilon h}\big|_{\epsilon=0}$. Therefore, for smooth functionals, D is a directional derivative. It is also routine to prove that if F, G are smooth functionals then, $D(FG) = FDG + GDF$.

Our next aim is to prove that D is *closable* as an *operator* from $L^p(\Omega)$ to $L^p(\Omega; H)$, for any $p \geq 1$. That is, if $\{F_n, n \geq 1\} \subset \mathcal{S}$ is a sequence converging to zero in $L^p(\Omega)$ and the sequence $\{DF_n, n \geq 1\}$ converges to G in $L^p(\Omega; H)$, then $G = 0$. The tool for this is a simple version of the ***integration by parts formula***.

Lemma 3.2

For any $F \in \mathcal{S}$, $h \in H$, we have

$$E\Big(\big\langle DF, h\big\rangle_H\Big) = E\big(FW(h)\big). \tag{3.9}$$

PROOF

Without loss of generality, we shall assume that

$$F = f\big(W(h_1), \dots, W(h_n)\big),$$

where $h_1, \dots, h_n$ are orthonormal elements of H and $h_1 = h$. Then

$$\begin{aligned} E\Big(\big\langle DF, h\big\rangle_H\Big) &= \int_{\mathbb{R}^n} \partial_1 f(x)\mu_n(dx) \\ &= \int_{\mathbb{R}^n} f(x)x_1\mu_n(dx) = E\big(FW(h_1)\big). \end{aligned}$$

The proof is complete. □

Let $F, G \in \mathcal{S}$. Applying formula (3.9) to the smooth functional FG yields

$$E\Big(G\big\langle DF, h\big\rangle_H\Big) = -E\Big(F\big\langle DG, h\big\rangle_H\Big) + E\big(FGW(h)\big). \tag{3.10}$$

With this result, we can now prove that D is closable. Indeed, consider a sequence $\{F_n, n \geq 1\} \subset \mathcal{S}$ satisfying the properties stated above. Let $h \in H$ and $F \in \mathcal{S}_b$ such that $FW(h)$ is bounded. Using (3.10), we obtain

$$E\Big(F\langle G, h\rangle_H\Big) = \lim_{n\to\infty} E\Big(F\langle DF_n, h\rangle_H\Big) = \lim_{n\to\infty} E\Big(-F_n\langle DF, h\rangle_H + F_n FW(h)\Big) = 0.$$

Indeed, the sequence $(F_n, n \geq 1)$ converges to zero in L^p and $\langle DF, h\rangle_H$, $FW(h)$ are bounded. This yields $G = 0$. □

Let $\tilde{\mathbb{D}}^{1,p}$ be the closure of the set $\mathcal{S}$ with respect to the seminorm

$$\|F\|'_{1,p} = \Big(E\big(|F|^p\big) + E\big(\|DF\|_H^p\big)\Big)^{\frac{1}{p}}. \tag{3.11}$$

The set $\tilde{\mathbb{D}}^{1,p}$ is the domain of the operator D in $L^p(\Omega)$. Notice that $\tilde{\mathbb{D}}^{1,p}$ is dense in $L^p(\Omega)$. The above procedure can be iterated as follows. Clearly, one can recursively define the operator D^k, $k \in \mathbb{N}$, on the set $\mathcal{S}$. This yields an $H^{\otimes k}$-valued random vector. As for D, one proves that D^k is closable. Then we can introduce the seminorms

$$\|F\|'_{k,p} = \left(E\big(|F|^p\big) + \sum_{j=1}^{k} E\Big(\big\|D^j F\big\|^p_{H^{\otimes j}}\Big)\right)^{\frac{1}{p}}, \tag{3.12}$$

$p \in [1, \infty)$, and define the sets $\tilde{\mathbb{D}}^{k,p}$ to be the closure of $\mathcal{S}$ with respect to the seminorm (3.12). Notice that by definition, $\tilde{\mathbb{D}}^{j,q} \subset \tilde{\mathbb{D}}^{k,p}$ for $k \leq j$ and $p \leq q$. A natural question is whether the spaces $\tilde{\mathbb{D}}^{k,p}$ and $\mathbb{D}^{k,p}$, defined in Section 3.1 by means of the operator L, do coincide. The answer is positive. This fact is a consequence of *Meyer's inequalities* —a deep mathematical result proved by Meyer in refence [37]. For the sake of completeness, we quote here this result without proof. The reader interested in the details is referred to reference [37] (see also ref. [44]).

Theorem 3.1

Let $p \in [1, \infty)$, $k \in \mathbb{N}$. There exist positive constants $c_{k,p}$, $C_{k,p}$, such that for any $F \in \mathcal{S}$,

$$c_{k,p} E\Big(\big\|D^k F\big\|^p_{H^{\otimes k}}\Big) \leq \|F\|^p_{k,p} \leq C_{k,p}\big(\|F\|'_{k,p}\big)^p. \tag{3.13}$$

Our next purpose is to determine the action of the operator D on each Wiener chaos. First let us make an observation. The Wiener chaos expansion developed in Section 3.1 can be extended to the more general setting of $L^2(\Omega; V)$, where V is a Hilbert space. Indeed, the following holds:

$$L^2(\Omega; V) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n(V),$$

with $\mathcal{H}_n(V) = \mathcal{H}_n \otimes V$.

Proposition 3.3

A random variable $F \in L^2(\Omega)$ belongs to $\mathbb{D}^{1,2}$ if and only if $\sum_{n=1}^{\infty} n \|J_n F\|_2^2 < \infty$. In this case

$$D(J_n(F)) = J_{n-1}(DF)$$

and

$$E\big(\|DF\|_H^2\big) = \sum_{n=1}^{\infty} n \|J_n F\|_2^2.$$

Proof

Consider the *multiple Hermite polynomials* H_a defined in (3.2). Then,

$$DH_a = \sqrt{a!} \sum_{j=1}^{\infty} \prod_{j \neq i} H_{a_i}\big(W(e_i)\big) H_{a_j - 1}\big(W(e_j)\big) e_j,$$

because $H_n' = H_{n-1}$. Notice that if $|a| = n$ then $DH_a \in \mathcal{H}_{n-1}(H)$. Moreover,

$$E\Big(\|DH_a\|_H^2\Big) = \sum_{j=1}^{\infty} \frac{a!}{(\prod_{j \neq i} a_i)!(a_j - 1)!} = \sum_{j=1}^{\infty} a_j = |a|.$$

This proves the result for $F = H_a$, which suffices to finish the proof. □

The following extension of the spaces $\mathbb{D}^{k,p}$ will be needed subsequently. Let $\mathcal{S}_V$ be the set of smooth random vectors taking values in V of the form

$$F = \sum_{j=1}^{n} F_j v_j, \qquad v_j \in V,\ F_j \in \mathcal{S},\ j = 1, \ldots, n.$$

By definition, the k-th derivative of F is given by $D^kF = \sum_{j=1}^n D^kF_j \otimes v_j$. As before, one can prove that D^k is a closable operator from $\mathcal{S}_V \subset L^p(\Omega; V)$ into $L^p(\Omega; H^{\otimes k} \otimes V)$, $p \geq 1$. Then, for any $k \in \mathbb{N}$, $p \in [1, \infty)$, we introduce the seminorms on $\mathcal{S}_V$ given by

$$\|F\|_{k,p,V}^p = E\big(\|F\|_V^p\big) + \sum_{j=1}^k E\Big(\big\|D^jF\big\|_{H^{\otimes j} \otimes V}^p\Big).$$

Then $\mathbb{D}^{k,p}(V)$ is the completion of the set $\mathcal{S}_V$ with respect to this norm. We define $\mathbb{D}^\infty = \bigcap_{k\geq 1} \bigcap_{p \geq 1} \mathbb{D}^{k,p}(V)$.

3.3 The integral or divergence operator

We introduce in this section an operator which plays a fundamental role in establishing the criteria for existence and uniqueness of densities for random vectors. As shall be explained later on, it corresponds to the infinite dimensional analogue of the operator δ_m defined in (2.11).

The *Malliavin derivative* D introduced in the previous section is an unbounded operator from $L^2(\Omega)$ into $L^2(\Omega; H)$. Moreover, the domain of D in $L^2(\Omega)$, $\mathbb{D}^{1,2}$, is dense in $L^2(\Omega)$. Then, by an standard procedure (see for instance ref. [69]) one can define its *adjoint* δ.

Indeed, the domain of the adjoint, denoted by $\operatorname{Dom}\delta$, is the set of random vectors $u \in L^2(\Omega; H)$ such that for any $F \in \mathbb{D}^{1,2}$,

$$\Big|E\big(\langle DF, u\rangle_H\big)\Big| \leq c\|F\|_2,$$

where c is a constant depending on u. If $u \in \operatorname{Dom}\delta$, then δu is the element of $L^2(\Omega)$ characterized by the identity

$$E\big(F\delta(u)\big) = E\big(\langle DF, u\rangle_H\big), \tag{3.14}$$

for all $F \in \mathbb{D}^{1,2}$.

Equation (3.14) expresses the duality between D and δ. It is called the ***integration by parts formula***. The analogy between δ and δ_m defined in (2.11) can be easily established on *finite dimensional* random vectors of $L^2(\Omega; H)$, as follows.

Let $\mathcal{S}_{\mathcal{H}}$ be the set of random vectors of the type

$$u = \sum_{j=1}^n F_j h_j,$$

where $F_j \in \mathcal{S}$, $h_j \in H$, $j = 1, \dots, n$. Let us prove that $u \in \operatorname{Dom}\delta$.

Indeed, owing to formula (3.10), for any $F \in \mathcal{S}$,

$$\begin{aligned}\Big|E\big(\langle DF, u\rangle_H\big)\Big| &= \left|\sum_{j=1}^{n} E\Big(F_j\big\langle DF, h_j\big\rangle_H\Big)\right| \\ &\le \sum_{j=1}^{n}\Big(\Big|E\Big(F\big\langle DF_j, h_j\big\rangle_H\Big)\Big| + \Big|E\big(FF_jW(h_j)\big)\Big|\Big) \\ &\le C\|F\|_2.\end{aligned}$$

Hence $u \in \operatorname{Dom}\delta$. Moreover, by the same computations,

$$\delta(u) = \sum_{j=1}^{n} F_jW(h_j) - \sum_{j=1}^{n}\big\langle DF_j, h_j\big\rangle_H. \tag{3.15}$$

Hence, the gradient operator in the finite dimensional case is replaced by the Malliavin *directional* derivative, and the coordinate variables x_i by the *random coordinates* $W(h_j)$.

We have seen in Proposition 3.3 that the operator D decreases by one the Wiener chaos order. Its adjoint, δ, does the opposite. We shall come back to this fact later.

Remark 3.3 The divergence operator or *Skorohod integral* coincides with a stochastic integral introduced by Skorohod in reference [62]. One of the interesting features of this integral is that it allows non-adapted integrands. We shall see in the next chapter that it is actually an extension of Itô's integral.

3.4 Differential calculus

In this section we prove several basic calculus rules based on the three operators defined so far. The first result is a *chain rule*.

Proposition 3.4

Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives. Let $F = (F^1, \dots, F^m)$ be a random vector whose components belong to $\mathbb{D}^{1,p}$ for some $p \ge 1$. Then $\varphi(F) \in \mathbb{D}^{1,p}$ and

$$D\big(\varphi(F)\big) = \sum_{i=1}^{m} \partial_i\varphi(F)DF^i. \tag{3.16}$$

The proof of this result is straightforward. First, we assume that $F \in \mathcal{S}$; in this case, formula (3.16) follows by the classical rules of differential calculus. The proof for $F \in \mathbb{D}^{1,p}$ is done by an approximation procedure.

The preceding chain rule can be improved to φ Lipschitz. The tool for this extension is given in the next Proposition.

Proposition 3.5

Let $(F_n, n \geq 1)$ be a sequence of random variables in $\mathbb{D}^{1,2}$ converging to F in $L^2(\Omega)$ and such that

$$\sup_n E\Big(\big\|DF_n\big\|_H^2\Big) < \infty. \tag{3.17}$$

Then F belongs to $\mathbb{D}^{1,2}$ and the sequence of derivatives $(DF_n, n \geq 1)$ converges to DF in the weak topology of $L^2(\Omega; H)$.

Proof

The assumption (3.17) yields the existence of a subsequence $(F_{n_k}, k \geq 1)$ such that the corresponding sequence of derivatives $(DF_{n_k}, k \geq 1)$ converges in the weak topology of $L^2(\Omega; H)$ to some element $\eta \in L^2(\Omega; H)$. In particular, for any $G \in L^2(\Omega; H)$, $\lim_{k\to\infty} E\big(\langle DF_{n_k}, J_l G\rangle_H\big) = E\big(\langle \eta, J_l G\rangle_H\big)$, where J_l denotes the projection on the l-th Wiener chaos $\mathcal{H}_l \otimes H$, $l \geq 0$.

The integration by parts formula and the convergence of the sequence $(F_n, n \geq 1)$ yield

$$\begin{aligned}\lim_{k\to\infty} E\Big(\big\langle DF_{n_k}, J_l G\big\rangle_H\Big) &= \lim_{k\to\infty} E\big(F_{n_k}\delta(J_l G)\big)\\ &= E\big(F\delta(J_l G)\big) = E\Big(\big\langle DF, J_l G\big\rangle_H\Big).\end{aligned}$$

Hence, every weakly convergent subsequence of $DF_n, n \geq 1$, must converge to the same limit and the whole sequence converges. Moreover, the random vectors η and DF have the same projection on each Wiener chaos; consequently, $\eta = DF$ as elements of $L^2(\Omega; H)$. □

Proposition 3.6

Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a globally ***Lipschitz function*** *and $F = (F^1, \dots, F^m)$ be a random vector with components in $\mathbb{D}^{1,2}$. Then*

$\varphi(F) \in \mathbb{D}^{1,2}$. *Moreover, there exists a bounded random vector* $G = (G_1, \dots, G_m)$ *such that*

$$D\big(\varphi(F)\big) = \sum_{i=1}^{m} G_i DF^i. \tag{3.18}$$

PROOF

The idea of the proof is as follows. First we regularize the function φ by *convolution* with an *approximation of the identity*. We apply Proposition 3.4 to the sequence obtained this way. Then we conclude by means of Proposition 3.5.

Let $\alpha \in \mathcal{C}_0^\infty(\mathbb{R}^m)$ be non-negative, with compact support and $\int_{\mathbb{R}^m} \alpha(x)dx = 1$. Define $\alpha_n(x) = n^m \alpha(nx)$ and $\varphi_n = \varphi * \alpha_n$. It is well known that $\varphi_n \in \mathcal{C}^\infty$ and that the sequence $(\varphi_n, n \geq 1)$ converges to φ uniformly. In addition $\nabla \varphi_n$ is bounded by the Lipschitz constant of φ.

By Proposition 3.4,

$$D\big(\varphi_n(F)\big) = \sum_{i=1}^{m} \partial_i \varphi_n(F) DF^i. \tag{3.19}$$

Now we apply Proposition 3.5 to the sequence $F_n = \varphi_n(F)$. It is clear that $\lim_{n\to\infty} \varphi_n(F) = \varphi(F)$ in $L^2(\Omega)$. Moreover, by the boundedness property on $\nabla\varphi_n$, the sequence $D\big(\varphi_n(F)\big)$, $n \geq 1$, is bounded in $L^2(\Omega; H)$. Hence $\varphi(F) \in \mathbb{D}^{1,2}$ and $D\big(\varphi_n(F)\big)$, $n \geq 1$ converges in the weak topology of $L^2(\Omega; H)$ to $D\big(\varphi(F)\big)$. Since the sequence $\nabla\varphi_n(F)$, $n \geq 1$, is bounded, a.s., there exists a subsequence that converges to some random bounded vector G in the weak topology of $L^2(\Omega; H)$. Passing to the limit in (3.19), we finish the proof of the Proposition. □

REMARK 3.4 Let $\varphi \in \mathcal{C}_p^\infty(\mathbb{R}^m)$ and $F = (F^1, \dots, F^m)$ be a random vector whose components belong to $\bigcap_{p\in[1,\infty)} \mathbb{D}^{1,p}$. Then the conclusion of Proposition 3.4 also holds. Moreover, $\varphi(F) \in \bigcap_{p\in[1,\infty)} \mathbb{D}^{1,p}$.

The chain rule (3.16) can be iterated; we obtain *Leibniz's rule* for Malliavin's derivatives. For example, if F is one-dimensional then

$$D^k\big(\varphi(F)\big) = \sum_{\ell=1}^{k} \sum_{\mathcal{P}_\ell} c_\ell \varphi^{(\ell)}(F) \prod_{i=1}^{\ell} D^{|p_i|} F, \tag{3.20}$$

where $\mathcal{P}_\ell$ denotes the set of partitions of $\{1,\dots,k\}$ consisting of ℓ disjoint sets $p_1,\dots,p_\ell$, $\ell=1,\dots,k$, $|p_i|$ denotes the cardinal of the set p_i and c_ℓ are positive coefficients.

For any $F\in \mathrm{Dom}D$, $h\in H$ we set $D_hF=\langle DF,h\rangle_H$. The next propositions give important calculus rules.

Proposition 3.7

Let $u\in\mathcal{S}_H$. Then

$$D_h\big(\delta(u)\big)=\langle u,h\rangle_H+\delta(D_hu). \tag{3.21}$$

Proof

Fix $u=\sum_{j=1}^n F_jh_j$, $F_j\in\mathcal{S}$, $h_j\in H$, $j=1,\dots,n$. By virtue of (3.15), we have

$$D_h\big(\delta(u)\big)=\sum_{j=1}^n\Big((D_hF_j)W(h_j)+F_j\big\langle h_j,h\big\rangle-\big\langle D(D_hF_j),h_j\big\rangle_H\Big).$$

Notice that by (3.15),

$$\delta(D_hu)=\sum_{j=1}^n\Big((D_hF_j)W(h_j)-\big\langle D(D_hF_j),h_j\big\rangle_H\Big). \tag{3.22}$$

Hence (3.21) holds. □

Proposition 3.8

Let $u,v\in\mathbb{D}^{1,2}(H)$. Then

$$E\big(\delta(u)\delta(v)\big)=E\big(\langle u,v\rangle_H\big)+E\big(\mathrm{tr}\,(Du\circ Dv)\big), \tag{3.23}$$

where $\mathrm{tr}\,(Du\circ Dv)=\sum_{i,j=1}^\infty D_{e_j}\big\langle u,e_i\big\rangle_H D_{e_i}\big\langle v,e_j\big\rangle_H$, *with* $(e_i,i\ge 1)$ *a complete orthonormal system in H.*

Consequently, if $u\in\mathbb{D}^{1,2}(H)$ then, $u\in\mathrm{Dom}\,\delta$ and

$$E\big(\delta(u)\big)^2=E(\|u\|_H^2)+E\big(\|Du\|_{H\otimes H}^2\big). \tag{3.24}$$

Proof

Assume first that $u, v \in \mathcal{S}_H$. The duality relation between D and δ yields

$$E\big(\delta(u)\delta(v)\big) = E\Big(\Big\langle v, D\big(\delta(u)\big)\Big\rangle_H\Big) = E\left(\sum_{i=1}^{\infty}\langle v, e_i\rangle_H D_{e_i}\big(\delta(u)\big)\right).$$

Owing to (3.21), this last expression is equal to

$$E\left(\sum_{i=1}^{\infty}\langle v, e_i\rangle_H\Big(\langle u, e_i\rangle_H + \delta(D_{e_i}u)\Big)\right).$$

The duality relation between D and δ implies

$$\begin{aligned}
E\Big(\langle v, e_i\rangle_H \delta(D_{e_i}u)\Big) &= E\Big(\Big\langle D_{e_i}u,\ D\langle v, e_i\rangle_H\Big\rangle_H\Big)\\
&= \sum_{j=1}^{\infty} E\Big(\Big\langle D_{e_i}\langle u, e_j\rangle_H e_j,\ D\langle v, e_i\rangle_H\Big\rangle\Big)\\
&= \sum_{j=1}^{\infty} E\Big(D_{e_i}\langle u, e_j\rangle_H D_{e_j}\langle v, e_i\rangle_H\Big).
\end{aligned}$$

This establishes (3.23) and for $u = v$ this implies (3.24).

The extension to $u, v \in \mathbb{D}^{1,2}(H)$ is done by a limit procedure. □

Remark 3.5 Proposition 3.8 can be used to extend the validity of (3.21) to $u \in \mathbb{D}^{2,2}(H)$. Indeed, let $u_n \in \mathcal{S}_H$ be a sequence of processes converging to u in $\mathbb{D}^{2,2}(H)$. Formula (3.21) holds true for u_n. We can take limits in $L^2(\Omega; H)$ as n tends to infinity and conclude, because the operators D and δ are closed.

Proposition 3.9

Let $F \in \mathbb{D}^{1,2}$, $u \in \operatorname{Dom}\delta$, $Fu \in L^2(\Omega; H)$. If $F\delta(u) - \langle DF, u\rangle_H \in L^2(\Omega)$, then

$$\delta(Fu) = F\delta(u) - \langle DF, u\rangle_H. \tag{3.25}$$

Proof

Assume first that $F \in \mathcal{S}$ and $u \in \mathcal{S}_H$. Let $G \in \mathcal{S}$. Then by the duality relation between D and δ and the calculus rules on the derivatives, we have

$$\begin{aligned} E\big(G\delta(Fu)\big) &= E\big(\langle DG, Fu\rangle_H\big) \\ &= E\Big(\Big\langle u, \big(D(FG) - GDF\big)\Big\rangle_H\Big) \\ &= E\Big(G\Big(F\delta(u) - \langle u, DF\rangle_H\Big)\Big). \end{aligned}$$

By the definition of the operator δ, (3.25) holds under the assumptions of the proposition. □

Proposition 3.10

Let $F \in L^2(\Omega)$. Then $F \in \operatorname{Dom} L$ if and only if $F \in \mathbb{D}^{1,2}$ and $DF \in \operatorname{Dom}\delta$. In this case

$$\delta(DF) = -LF. \tag{3.26}$$

Proof

Assume first that $F \in \operatorname{Dom} D$ and $DF \in \operatorname{Dom}\delta$. Let $G \in \mathcal{S}$. Using the duality relation, Proposition 3.3 and (3.5) we obtain

$$\begin{aligned} E\big(G\delta(DF)\big) = E\big(\langle DG, DF\rangle_H\big) &= \sum_{n=0}^{\infty} nE(J_n G J_n F) \\ &= E\left(G\sum_{n=0}^{\infty} nJ_n F\right) = -E(GLF). \end{aligned}$$

Thus, $F \in \operatorname{Dom} L$ and $LF = -\delta(DF)$.

Reciprocally, assume that $F \in \operatorname{Dom} L$. Then $\sum_{n=1}^{\infty} n^2 \big\|J_n F\big\|_2^2 < \infty$. Consequently, by Proposition 3.3 $F \in \operatorname{Dom} D$ and, by the previous computations

$$E(GLF) = -E\big(\langle DG, DF\rangle_H\big).$$

Moreover,

$$\big|E(GLF)\big| \le \|G\|_2 \, \|LF\|_2 < \infty.$$

Consequently, $DF \in \operatorname{Dom}\delta$.

This finishes the proof. □

As a consequence of this proposition and Proposition 3.3, the divergence operator increases by one degree the Wiener chaos order. The next result shows that L is a kind of *second order differential operator*.

Proposition 3.11

Let $F = (F^1, \dots, F^m)$ be a random vector with components in $\mathbb{D}^{2,4}$. Let $\varphi \in \mathcal{C}^2(\mathbb{R}^m)$ with bounded partial derivatives up to the second order. Then $\varphi(F) \in \operatorname{Dom} L$ and

$$L\big(\varphi(F)\big) = \sum_{i,j=1}^{m} (\partial_{i,j}^2 \varphi)(F) \big\langle DF^i, DF^j \big\rangle_H + \sum_{i=1}^{m} (\partial_i \varphi)(F) LF^i. \quad (3.27)$$

Proof

For the sake of simplicity, we shall give the proof for $m = 1$. Suppose that $F \in \mathcal{S}$, $F = f\big(W(h_1), \dots, W(h_n)\big)$. Then, by virtue of Proposition 3.10, (3.16), (3.8) and (3.15) we obtain

$$\begin{aligned}
L\varphi(F) &= -\delta\Big(D\big(\varphi(F)\big)\Big) = -\delta\big(\varphi'(F) DF\big) \\
&= -\delta\left(\varphi'(F) \sum_{i=1}^{n} \partial_i f\big(W(h_1), \dots, W(h_n)\big) h_i\right) \\
&= -\delta\left(\sum_{i=1}^{n} \varphi'\Big(f\big(W(h_1), \dots, W(h_n)\big)\Big) \partial_i f\big(W(h_1), \dots, W(h_n)\big) h_i\right) \\
&= -\sum_{i=1}^{n} \partial_i (\varphi \circ f)\big(W(h_1), \dots, W(h_n)\big) W(h_i) \\
&\quad + \sum_{i,j=1}^{n} \partial_{i,j} (\varphi \circ f)\big(W(h_1), \dots, W(h_n)\big) \langle h_i . h_j \rangle.
\end{aligned}$$

Hence (3.27) holds for smooth random variables. The extension to $F \in \mathbb{D}^{2,4}$ follows by a standard approximation procedure. □

3.5 Calculus with multiple Wiener integrals

In the previous sections, we have introduced three fundamental operators defined on spaces related with a Gaussian space. We have given their actions on any Wiener chaos. In this section, we aim to go further in this direction when considering the special case $H = L^2(A, \mathcal{A}, m)$. The particular feature of this example is that the Wiener chaos can be described in terms of stochastic integrals —the Itô multiple stochastic integrals. Therefore, the action of the operators and their domains can be described in terms of conditions on these integrals. We gain in operativeness because additional tools of stochastic calculus become available. For the sake of completeness we start with a very short account on multiple Itô-Wiener integrals and their role in the Wiener chaos decomposition. For complete details on the topic we refer the reader to the original work by Itô [20] (see also ref. [43]).

The framework here consists of a separable σ-finite measure space $(A, \mathcal{A}, m)$, the Hilbert space $H = L^2(A, \mathcal{A}, m)$ and the white noise $W = (W(F), F \in \mathcal{A})$ based on m. We assume that the measure m has no atoms.

The *multiple Itô-Wiener integrals* are defined as follows. Let $\mathcal{E}_n$ be the set of deterministic *elementary functions* of the type

$$f(t_1, \dots, t_n) = \sum_{j_1,\dots,j_n=1}^{k} a_{j_1,\dots,j_n} \mathbf{1}_{A_{j_1} \times \cdots \times A_{j_n}}(t_1, \dots, t_n),$$

where $A_{j_1}, \dots, A_{j_n}$ are pairwise-disjoint elements of $\mathcal{A}$ with finite measure; the coefficients $a_{j_1,\dots,j_n}$ vanish whenever two of the indices $j_1, \dots, j_n$ coincide.

For this class of functions, we define

$$I_n(f) = \sum_{j_1,\dots,j_n=1}^{k} a_{j_1,\dots,j_n} W(A_{j_1}) \cdots W(A_{j_n}).$$

For any function f defined on A^n, we denote by $\tilde{f}$ its symmetrization. I_n defines a linear map from $\mathcal{E}_n$ into $L^2(\Omega)$ which satisfies the following properties:

1) $I_n(f) = I_n(\tilde{f})$,

2) $E\big(I_n(f) I_m(g)\big) = \begin{cases} 0 & \text{if } n \neq m, \\ n! \langle \tilde{f}, \tilde{g} \rangle_{L^2(A^m)} & \text{if } n = m. \end{cases}$

The set $\mathcal{E}_n$ is dense in $L^2(A^n)$ and I_n extends to a linear continuous functional, defined on $L^2(A^n)$, taking values in $L^2(\Omega)$.

Assume that $A = \mathbb{R}_+$, $\mathcal{A}$ is the corresponding Borel set and m is the Lebesgue measure. Let $f \in L^2(A^n)$ be a symmetric function. The multiple Itô-Wiener integral $I_n(f)$ coincides in this case with an iterated Itô integral. That is,

$$I_n(f) = n! \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \dots, t_n)\, dW_{t_1} \cdots dW_{t_n}. \tag{3.28}$$

Indeed, this is clear for elementary functions of the type described above, and for general f, we use a density argument. Notice that Itô's integral satisfies the same isometry property as the multiple Itô-Wiener integral.

One of the basic results in Itô's paper states that the n-th Wiener chaos $\mathcal{H}_n$ coincides with the image under I_n of $L^2(A^n)$. That is, for any $F \in L^2(\Omega)$, the projection $J_n(F)$ can be written as $I_n(f_n)$, for some $f_n \in L^2(A^n)$. This leads to the following form of the Wiener chaos decomposition:

$$F = E(F) + \sum_{n=1}^{\infty} I_n(f_n), \tag{3.29}$$

with $f_n \in L^2(A^n)$ symmetric and uniquely determined by F.

Owing to (3.5) and (3.29), we have the following result:

Proposition 3.12

A random vector $F \in L^2(\Omega)$ belongs to the domain of L if and only if

$$\sum_{n=1}^{\infty} n^2 n! \big\|\tilde{f}_n\big\|^2_{L^2(A^m)} < \infty,$$

and in this case,

$$LF = \sum_{n=1}^{\infty} -nI_n(f_n).$$

The corresponding result for the derivative operator is as follows:

Proposition 3.13

A random vector $F \in L^2(\Omega)$ belongs to the domain of D if and only if

$$\sum_{n=1}^{\infty} n n! \|\tilde{f}_n\|^2_{L^2(A^m)} < \infty,$$

and in this case,

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}\big(f_n(\cdot, t)\big), \tag{3.30}$$

for all $t \in A$.

Proof

The characterization of the domain follows trivially from Proposition 3.3. Hence only (3.30) must be checked. Clearly, it suffices to prove that

$$D_t I_n(f_n) = n I_{n-1}\big(f_n(\cdot, t)\big). \tag{3.31}$$

Assume first that f_n is an elementary symmetric function. Then

$$\begin{aligned} D_t I_n(f_n) &= \sum_{l=1}^{n} \sum_{j_1,\dots,j_n=1}^{k} a_{j_1,\dots,j_n} W(A_{j_1}) \cdots \mathbf{1}_{A_{j_l}}(t) \cdots W(A_{j_n}) \\ &= n I_{n-1}\big(f_n(\cdot, t)\big). \end{aligned}$$

For a general $f_n \in L^2(A^n)$, the result follows easily by an obvious approximation argument. □

We recall that for $F \in \mathbb{D}^{1,2}$, the derivative DF belongs to $L^2(\Omega; H)$. In the setting of this section $L^2(\Omega; H) \simeq L^2(\Omega \times A)$; thus DF is a function of two variables, $\omega \in \Omega$ and $t \in A$. As usual, we shall not write the dependence on ω. We note $DF(t) = D_t F$.

Finally we study the *divergence operator*.

Proposition 3.14

Let $u \in L^2(\Omega \times A)$ with Wiener chaos decomposition

$$u(t) = \sum_{n=0}^{\infty} I_n\big(f_n(\cdot, t)\big).$$

We assume that for each $n \geq 1$, $f_n \in L^2(A^{n+1})$ is a symmetric function in the first n variables. Then u belongs to $\mathrm{Dom}\,\delta$ *if and only if the series*

$$\sum_{n=0}^{\infty} I_{n+1}(f_n) \tag{3.32}$$

converges in $L^2(\Omega)$, and in this case

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(f_n) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n), \tag{3.33}$$

where $\tilde{f}_n$ denotes the symmetrization of f_n in its $n+1$ variables.

Proof

It is based on the duality relation between D and δ. Let $F = I_n(f)$, with f symmetric. The orthogonality of the Itô integrals yields

$$\begin{aligned} E\big(\langle u, DF\rangle_H\big) &= \int_A E\Big(I_{n-1}\big(f_{n-1}(\cdot,t)\big) n I_{n-1}\big(f(\cdot,t)\big)\Big) m(dt) \\ &= n(n-1)! \int_A \big\langle f_{n-1}(\cdot,t), f(\cdot,t)\big\rangle_{L^2(A^{n-1})} m(dt) \\ &= n! \big\langle f_{n-1}, f\big\rangle_{L^2(A^n)} = n! \big\langle \tilde{f}_{n-1}, f\big\rangle_{L^2(A^n)} \\ &= E\big(I_n(\tilde{f}_{n-1}) I_n(f)\big) = E\big(I_n(\tilde{f}_{n-1}) F\big). \end{aligned}$$

Assume that u belongs to $\mathrm{Dom}\,\delta$. The preceding equalities show that on the Wiener chaos of order n, $n \geq 1$, $\delta(u) = I_n(\tilde{f}_{n-1})$. Thus the series (3.32) converges in $L^2(\Omega)$ and (3.33) holds true.

Assume now that the series (3.32) converges. Then, by the arguments above, we obtain

$$\Big|E\big(\langle u, DF\rangle_H\big)\Big| \leq \|F\|_2 \left\| \sum_{n=1}^{\infty} I_n(\tilde{f}_{n-1}) \right\|_2 \leq C\|F\|_2.$$

Hence $u \in \mathrm{Dom}\,\delta$ and the formula (3.33) holds. □

We mentioned in Remark 3.5 that the formula (3.21) can be extended to random vectors $u \in \mathbb{D}^{2,2}(H)$. We are going to show that, in the context of this section, (3.21) holds in the less restrictive situation $u \in \mathbb{D}^{1,2}(H)$. This is the goal of the next statement.

Proposition 3.15

Let $u \in \mathbb{D}^{1,2}(H)$. Assume that for almost every $t \in A$, the process $(D_t u(s), s \in A)$ belongs to Dom δ *and there is a version of the process $\big(\delta(D_t u(s)), t \in A\big)$ which is in $L^2(\Omega \times A)$. Then δu belongs to $\mathbb{D}^{1,2}$ and*

$$D_t\big(\delta(u)\big) = u(t) + \delta(D_t u), \tag{3.34}$$

for all $t \in A$.

Proof

Let $u(t) = \sum_{n=0}^{\infty} I_n\big(f_n(\cdot, t)\big)$. Then Propositions 3.13 and 3.14 yield

$$\begin{aligned} D_t\big(\delta(u)\big) &= D_t\left(\sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n)\right) = \sum_{n=0}^{\infty}(n+1) I_n\big(\tilde{f}_n(\cdot, t)\big) \\ &= u(t) + \sum_{n=0}^{\infty} I_n\left(\sum_{i=1}^{n} f_n(t_1, \dots, \hat{t}_i, \dots, t_n, t, t_i)\right). \end{aligned}$$

Moreover,

$$I_n\left(\sum_{i=1}^{n} f_n(t_1, \dots, \hat{t}_i, \dots, t_n, t, t_i)\right) = n I_n\big(\varphi_n(\cdot, t, \cdot)\big),$$

where $\varphi_n(\cdot, t, \cdot)$ denotes the symmetrization of the function

$$(t_1, \dots, t_n) \longrightarrow f_n(t_1, \dots, t_{n-1}, t, t_n).$$

Let us now compute $\delta(D_t u)$. By virtue of Propostions 3.13 and 3.14 we obtain

$$\begin{aligned} \delta(D_t u) &= \delta\left(\sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t, s)\right) \\ &= \sum_{n=1}^{\infty} n I_n\big(\varphi_n(\cdot, t, \cdot)\big). \end{aligned}$$

Thus the formula (3.34) is completely proved. □

3.6 Local property of the operators

In this section, we return to the general context described at the beginning of the chapter.

Let $A \in \mathcal{G}$. An operator $\mathcal{O}$ defined on some space of random variables possesses the ***local property*** if, for any random variable F such that $F = 0$ a.s. on A, one has $\mathcal{O}(F) = 0$ a.s. on A.

We shall prove that the derivative operator D has this property. By duality, the property transfers to the adjoint δ. Finally, Proposition 3.10 yields that L is also a local operator.

The results of this section will not be used very often in the remainder of this book. However, they deserve to be presented. The local property of these operators makes it possible to formulate weak versions of Malliavin criteria for existence and smoothness of density that are specially suitable for some classes of SPDE's.

Proposition 3.16

The derivative operator D has the local property on the space $\mathbb{D}^{1,1}$.

Proof

Let $F \in \mathbb{D}^{1,1} \cap L^\infty(\Omega)$ and $A \in \mathcal{G}$ be such that $F = 0$, a.s. on A. Consider a function $\varphi \in \mathcal{C}^\infty$, $\varphi \geq 0$, $\varphi(0) = 1$, with support included in $[-1, 1]$. Set $\varphi_\epsilon(x) = \varphi(\frac{x}{\epsilon})$, $\epsilon > 0$. Let

$$\Psi_\epsilon(x) = \int_{-\infty}^{x} \varphi_\epsilon(y)\, dy.$$

The chain rule yields $\Psi_\epsilon(F) \in \mathbb{D}^{1,1}$ and $D\Psi_\epsilon(F) = \varphi_\epsilon(F)DF$.

Let u be an H-valued random variable of the form

$$u = \sum_{j=1}^{n} F_j h_j,$$

with $F_j \in \mathcal{S}_b$. We notice that the duality relation between D and δ holds for $F \in \mathbb{D}^{1,1} \cap L^\infty(\Omega)$ and u of the kind described above. Moreover, u is total in $L^1(\Omega; H)$, that is, if $v \in L^1(\Omega; H)$ satisfies $E\big(\langle v, u\rangle_H\big) = 0$ for every u in the class, then $v = 0$. Therefore,

$$\begin{aligned}\Big|E\big(\varphi_\epsilon(F)\langle DF, u\rangle_H\big)\Big| &= \Big|E\Big(\Big\langle \big(D(\Psi_\epsilon(F)\big), u\Big\rangle_H\Big)\Big| \\ &= \Big|E\big(\Psi_\epsilon(F)\delta(u)\big)\Big| \leq \epsilon\|\varphi\|_\infty E\Big(\big|\delta(u)\big|\Big).\end{aligned}$$

Taking limits as ϵ tends to zero, we obtain

$$E\big(\mathbf{1}_{(F=0)}\langle DF, u\rangle_H\big) = 0.$$

Finally, we notice that by replacing F by $\arctan F$, the restriction $F \in L^\infty(\Omega)$ can be removed. Actually, instead of the function arctan one could take any bounded function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$. The proof is complete. □

We now study the corresponding property for the divergence operator.

Proposition 3.17

The operator δ is local on $\mathbb{D}^{1,2}(H)$.

Proof

Let φ be a function defined as in the previous proposition and F be a smooth functional of the form

$$F = f\big(W(h_1), \ldots, W(h_n)\big),$$

with $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Then for any $u \in \mathbb{D}^{1,2}(H)$, $F\varphi_\epsilon\big(\|u\|_H^2\big)$ belongs to $\mathbb{D}^{1,2}$. By duality,

$$\begin{aligned}
&E\Big(\delta(u) F\varphi_\epsilon\big(\|u\|_H^2\big)\Big)\\
&\qquad = E\Big(\Big\langle u, D\Big(F\varphi_\epsilon\big(\|u\|_H^2\big)\Big)\Big\rangle_H\Big)\\
&\qquad = E\Big(\varphi_\epsilon\big(\|u\|_H^2\big)\langle u, DF\rangle_H\Big) + 2E\Big(F\varphi'_\epsilon\big(\|u\|_H^2\big)\langle u, Du\rangle_H\Big).
\end{aligned}$$

Consider the random variable

$$X_\epsilon = \varphi_\epsilon\big(\|u\|_H^2\big)\langle u, DF\rangle_H + 2F\varphi'_\epsilon\big(\|u\|_H^2\big)\langle u, Du\rangle_H.$$

Assume that $u(\omega) = 0$, P a.s. on A. Then, as $\epsilon \to 0$, $X_\epsilon \to 0$ a.s. on A. Moreover,

$$\begin{aligned}
\Big|\varphi_\epsilon\big(\|u\|_H^2\big)\langle u, DF\rangle_H\Big| &\le \|\varphi\|_\infty \|u\|_H \|DF\|_H,\\
\Big|\varphi'_\epsilon\big(\|u\|_H^2\big)\langle u, Du\rangle_H\Big| &\le \sup_x \big|x\varphi'_\epsilon(x)\big| \|Du\|_{H\otimes H} \le \|\varphi'\|_\infty \|Du\|_{H\otimes H}.
\end{aligned}$$

Hence, by bounded convergence, we conclude. □

The local property of the operator D makes it possible to define localized versions of the domains of this operator in $L^p(\Omega)$.

Indeed, let V be a Hilbert space; we define $\mathbb{D}^{k,p}_{\text{loc}}(V)$ as the set of V-valued random vectors such that there exists an increasing sequence $\Omega_n \subset \Omega$, and a sequence $F_n \in \mathbb{D}^{k,p}(V)$, $n \geq 1$, such that

1) $\Omega_n \uparrow \Omega$, a.s.
2) $F_n = F$ on Ω_n.

For $F \in \mathbb{D}^{k,p}_{\text{loc}}(V)$, we define $DF = DF_n$, on Ω_n. The local property of D ensures that this is well defined. Analogously, if u is an element of $\mathbb{D}^{1,2}_{\text{loc}}(H)$, we define $\delta(u) = \delta(u_n)$ on Ω_n. Remember that $\mathbb{D}^{1,2}(H) \subset \operatorname{Dom}\delta$.

Comments

This chapter requires knowledge of Itô's results on multiple stochastic integrals and their role in the Wiener chaos decomposition. They are proved in reference [20]. One could entitle this chapter "Essentials of Malliavin Calculus". Indeed, it is a very brief account of a deep and large theory presented in a quite simplified way. Since the seminal work by Malliavin [33], there have been many contributions to understand and develop his ideas using different approaches. We would like to mention here some of them in the context of Gaussian spaces and in the form of courses and lecture notes. In chronological order they are refences [63], [68], [11], [9], [48], [19], [43], [65], [34] and [44].

In view of the applications to SPDE's with coloured noise, we have presented the basic notions of Malliavin Calculus in the general setting of a Gaussian process indexed by a Hilbert-valued parameter. Our main references are [68] and [44].

3.7 Exercises

3.7.1

Let $g \in L^2\big([0,T]\big)$ and set $\|g\| = \|g\|_{L^2([0,T])}$. Consider the random variable

$$X = \exp\left(\int_0^T g(s)\, dW(s) - \frac{1}{2}\, \|g\|^2\right),$$

where W is a standard *Wiener process*. Show that the projection of X on the n-th Wiener chaos is

$$J_n X = \|g\|^n H_n\left(\frac{\int_0^T g(s)\, dW_s}{\|g\|}\right).$$

3.7.2

Prove the following identities, where the symbol δ denotes the Skorohod integral.

1) $\displaystyle\int_0^T W(T)\delta W(t) = W^2(T) - T.$

2) $\displaystyle\int_0^T W(t)\big[W(T) - W(t)\big]\delta W(t) = \frac{1}{6}\big(W^3(T) - 3TW(T)\big).$

Hint: Apply the results of Section 3.5 in the particular case where $A = [0,T]$, $\mathcal{A}$ is the Borel σ-algebra of sets of $[0,T]$ and m is Lebesgue measure.

3.7.3

Consider the framework of Section 3.5. Let F be a random variable belonging to the domain of D^n, the n-th iterate of D. Prove Stroock's formula

$$f_n(t_1,\dots,t_n) = \frac{E(D^n_{t_1,\dots,t_n}F)}{n!}.$$

Hint: Apply recursively formula (3.30).

3.7.4

Let $(W_t, t\in[0,T])$ be a standard Wiener process. Find the Malliavin derivative of the following random variables:

1) $F = \exp(W_t)$,

2) $\displaystyle F = \int_0^T\left(\int_0^{t_2}\cos(t_1+t_2)\,dW(t_1)\right)dW(t_2),$

3) $F = 3W_sW_t^2 + \log(1+W_s^2)$,

4) $\displaystyle F = \int_0^T W_t\delta W_s,$

$s,t\in[0,T]$.

3.7.5

Let $(W_t, t\in[0,1])$ be a standard Wiener process. Prove that the random variable $X = \sup_{t\in[0,1]} W_t$ belongs to $\mathbb{D}^{1,2}$ and $D_tX = \mathbf{1}_{[0,\tau]}(t)$, where τ is the a.s. unique point where the maximum of W is attained.

Hint: Consider a countable dense subset of $[0,1]$, $(t_i, i\in\mathbb{N})$. Set

$$X_n = \max_{0\le i\le n} W_{t_i} \quad\text{and}\quad \varphi_n(x_1,\dots,x_n) = \max(x_1,\dots,x_n).$$

Prove that $\varphi_n(X_n)\in\mathbb{D}^{1,2}$ and apply Proposition 3.5.

3.7.6

Consider the Gaussian space associated with a standard Wiener process on $[0,1]$. Let $f(\omega,t) = \sum_{i=0}^{n-1} c_i \mathbf{1}_{A_i} \mathbf{1}_{]t_i,t_{i+1}]}(t)$, $u \in L^2\big([0,1],\mathbb{D}^{1,2}\big)$, $M_t = \delta(u\mathbf{1}_{[0,t]})$, $t \in [0,1]$. Prove that $fu \in \operatorname{Dom}\delta$ and

$$\delta(fu) = \sum_{i=0}^{n-1} c_i \mathbf{1}_{A_i}(M_{t_{i+1}} - M_{t_i}).$$

Hint: Use the definition of the adjoint operator δ together with an approximation for $\mathbf{1}_{A_i}$ (see ref. [22]).

Chapter 4

Representation of Wiener Functionals

This chapter is devoted to an application of the Malliavin calculus to integral representation of square integrable random variables. The starting point is the renowned result by Itô which we now quote.

Let $W = \big(W_t, t \in [0,T]\big)$ be a standard one-dimensional Wiener process. Consider a random variable F that is measurable with respect to the σ-field generated by W. Then there exists a measurable and adapted stochastic process Φ, satisfying

$$E\left(\int_0^T \Phi^2(t)\,dt\right) < \infty$$

and such that

$$F = E(F) + \int_0^T \Phi(t)\,dW_t. \tag{4.1}$$

We shall prove that if F has some regularity in the Malliavin sense, then the kernel process Φ admits a description in terms of the Malliavin derivative of F. We shall apply this kind of result to the analysis of portfolios in finance.

In the application of this technique, we face the following question: Is the *Itô integral* consistent with the Skorohod one? That is, if a process is integrable in the sense of Itô, does it belong to $\operatorname{Dom}\delta$, and do both integrals coincide? The first section of this chapter is devoted to this question.

4.1 The Itô integral and the divergence operator

For any $G \in \mathcal{A}$, we denote by $\mathcal{F}_G$ the σ-field generated by the random variables $W(B)$, $B \in \mathcal{A}$, $B \subset G$.

Lemma 4.1

Let W be a white noise based on $(A, \mathcal{A}, m)$. Let F be a square integrable random variable with Wiener chaos representation $F = \sum_{n=0}^{\infty} I_n(f_n)$. Then, for any $G \in \mathcal{A}$,

$$E(F \mid \mathcal{F}_G) = \sum_{n=0}^{\infty} I_n(f_n \mathbf{1}_G^{\otimes n}). \tag{4.2}$$

Proof

It suffices to prove the lemma for $F = I_n(f_n)$, where $f_n \in L^2(A^n)$ and is symmetric. Moreover, since the set $\mathcal{E}_n$ of *elementary functions* is dense in $L^2(A^n)$, we may assume that

$$f_n = \mathbf{1}_{B_1 \times \cdots \times B_n},$$

where $B_1, \ldots, B_n$ are mutually disjoint sets of $\mathcal{A}$ having finite m measure.

For this kind of F, we have

$$\begin{aligned} E(F \mid \mathcal{F}_G) &= E\big(W(B_1) \cdots W(B_n) \mid \mathcal{F}_G\big) \\ &= E\left(\prod_{i=1}^{n} \big(W(B_i \cap G) + W(B_i \cap G^c)\big) \mid \mathcal{F}_G\right) \\ &= I_n\big(\mathbf{1}_{(B_1 \cap G) \times \cdots \times (B_n \cap G)}\big), \end{aligned}$$

where the last equality holds because of independence. Therefore the lemma is proved. □

Lemma 4.2

Assume that $F \in \mathbb{D}^{1,2}$. Let $G \in \mathcal{A}$. Then $E(F \mid \mathcal{F}_G) \in \mathbb{D}^{1,2}$ and

$$D_t E(F \mid \mathcal{F}_G) = E(D_t F \mid \mathcal{F}_G) \mathbf{1}_G(t).$$

Hence, if in addition F is $\mathcal{F}_G$-measurable, then $D_t F$ vanishes almost everywhere in $\Omega \times G^c$.

Proof

By virtue of Lemma 4.1 and Proposition 3.13, we have

$$\begin{aligned} D_t E(F \mid \mathcal{F}_G) &= \sum_{n=1}^{\infty} n I_{n-1}\Big(f_n(\cdot,t)\,\mathbf{1}_G^{\otimes(n-1)}\Big)\,\mathbf{1}_G(t) \\ &= E(D_t F \mid \mathcal{F}_G)\,\mathbf{1}_G(t). \end{aligned}$$

If F is $\mathcal{F}_G$-measurable, then the preceding equality yields

$$D_t F = D_t F\,\mathbf{1}_G(t).$$

Hence $(D_t F)(\omega) = 0$ if $(\omega, t) \in \Omega \times G^c$. The lemma is proved. □

Lemma 4.3

Let $G \in \mathcal{A}$, $m(G) < \infty$. Let F be an $\mathcal{F}_{G^c}$-measurable random variable in $L^2(\Omega)$. Then the process $F\,\mathbf{1}_G$ belongs to $\operatorname{Dom}\delta$ *and*

$$\delta(F\,\mathbf{1}_G) = FW(G).$$

Proof

Assume first that $F \in \mathcal{S}$. Then, by (3.15), we have

$$\delta(F\,\mathbf{1}_G) = FW(G) - \int_A D_t F\,\mathbf{1}_G(t) m(dt).$$

By the preceding Lemma, $\int_A D_t F\,\mathbf{1}_G(t) m(dt) = 0$. Hence the result is true in this particular situation. Since $\mathcal{S}$ is dense in $L^2(\Omega)$ and δ is closed, the result extends to $F \in L^2(\Omega)$. □

Consider the particular case $H = L^2(A, \mathcal{A}, m)$, with $A = [0,T] \times \{1, \ldots, d\}$. That is, the noise W is a d-dimensional standard Wiener process. Let L^2_a be the class of measurable adapted processes belonging to $L^2\big(\Omega \times [0,T]; \mathbb{R}^d\big)$. For any $\Phi \in L^2_a$, the *Itô integral* $\int_0^T \Phi(t)\,dW_t$ is well defined (see for instance ref. [57]). We now establish its relationship with the Skorohod integral.

Proposition 4.1

We have $L^2_a \subset$ $\operatorname{Dom}\delta$ *and on L^2_a, the operator δ coincides with the Itô integral.*

PROOF

For the sake of simplicity, we shall assume that W is one-dimensional. Let $u \in L_a^2$ be a simple process:

$$u(t) = \sum_{j=1}^{m} F_j \, \mathbf{1}_{(t_j, t_{j+1}]}(t),$$

where the F_j are square integrable, $\mathcal{F}_{[0,t_j]}$-measurable random variables and $0 \le t_1 < \cdots < t_{m+1} \le T$.

Lemma 4.3 yields $u \in \operatorname{Dom} \delta$ and

$$\delta(u) = \sum_{j=1}^{m} F_j \big(W(t_{j+1}) - W(t_j) \big). \tag{4.3}$$

Therefore, for elementary processes the Itô and the Skorohod integrals coincide. The fundamental result in the construction of Itô's integrals says that any $u \in L_a^2$ is approximated in the norm of $L^2(\Omega \times [0,T])$ by a sequence u_n of elementary processes for which (4.3) holds and we have

$$\begin{aligned} \int_0^T u(t)\, dW_t &= L^2(\Omega) - \lim_{n \to \infty} \int_0^T u_n(t)\, dW_t \\ &= \lim_{n \to \infty} \delta(u_n). \end{aligned}$$

Since δ is closed, we conclude that $u \in \operatorname{Dom} \delta$ and $\delta(u) = \int_0^T u(t)\, dW_t$. □

4.2 The Clark-Ocone formula

The framework here is that of a white noise on $L^2\big([0,T], \mathcal{B}\big([0,T]\big), \ell\big)$, where ℓ denotes the Lebesgue measure. That is, the Gaussian process is a standard one-dimensional Wiener process W. We denote by $\mathcal{F}_t$ the σ-field $\mathcal{F}_{[0,t]}$, $t \in [0,T]$.

Theorem 4.1

For any random variable $F \in \mathbb{D}^{1,2}$,

$$F = E(F) + \int_0^T E(D_t F \mid \mathcal{F}_t)\, dW_t. \tag{4.4}$$

Proof

Owing to Proposition 3.13 and Lemma 4.1, we have

$$\begin{aligned} E(D_t F \mid \mathcal{F}_t) &= \sum_{n=1}^{\infty} nE\Big(I_{n-1}\big(f_n(\cdot, t)\big) \mid \mathcal{F}_t\Big) \\ &= \sum_{n=1}^{\infty} nI_{n-1}\big(f_n(t_1, \dots, t_{n-1}, t)\, \mathbf{1}_{(t_1 \vee \dots \vee t_{n-1} \le t)}\big). \end{aligned}$$

Let $u_t = E(D_t F \mid \mathcal{F}_t)$. By Proposition 3.14, the integral $\delta(u)$ is computed as follows:

$$\begin{aligned} \delta(u) &= \sum_{n=1}^{\infty} nI_n\big(f_n(t_1, \dots, t_{n-1}, t)\, \mathbf{1}_{(t_1 \vee \dots \vee t_{n-1} \le t)}\big)^s \\ &= \sum_{n=1}^{\infty} I_n(f_n) \\ &= F - E(F), \end{aligned}$$

where the superscript "s" means symmetrization in all the variables. Indeed, f_n is symmetric in its n variables and a simple computation shows that $\big(\mathbf{1}_{(t_1 \vee \dots \vee t_{n-1} \le t)}\big)^s = \frac{1}{n}$. Clearly, the process $\big(u_t = E(D_t F / \mathcal{F}_t),\ t \in [0,T]\big)$ belongs to L^2_a. Hence the integral $\delta(u)$ is an Itô integral. This proves (4.4). □

4.3 Generalized Clark-Ocone formula

In this section, we consider transformations of an m-dimensional Wiener process by means of a drift. More precisely, let $\big(\theta(t), t \in [0,T]\big)$ be an $\mathbb{R}^m$-valued process, $\mathcal{F}_t$-adapted and satisfying ***Novikov's condition***

$$E\left(\exp\left(\frac{1}{2}\int_0^T \theta^2(s)\, ds\right)\right) < \infty. \tag{4.5}$$

Set

$$Z(t) = \exp\left(-\int_0^t \theta(s)\, dW_s - \frac{1}{2}\int_0^t \theta^2(s)\, ds\right), \tag{4.6}$$

$$\tilde{W}(t) = \int_0^t \theta(s)\, ds + W(t), \tag{4.7}$$

where $0 \leq t \leq T$. Define a measure on $\mathcal{G} = \mathcal{F}_T$ by

$$dQ = Z(T)\, dP. \tag{4.8}$$

Girsanov's theorem states that $\tilde{W} = \big(\tilde{W}(t), t \in [0,T]\big)$ is a Wiener process with respect to the probability Q. In addition $\tilde{W}$ is an $\mathcal{F}_t$-martingale with respect to Q (see for instance refs. [23] and [50]).

The purpose is to obtain a representation result in the spirit of the previous Theorem 4.1 but with respect to the new Wiener process $\tilde{W}$. A motivation for this extension is the pricing of options in finance, as we shall see later.

In the sequel we shall write E for the expectation operator with respect to the probability P and E_Q for that with respect to Q defined by (4.8).

Remark 4.1 If F is $\mathcal{F}_t$-measurable then $D_sF = 0$ when $s > t$. This is a trivial consequence of Lemma 4.2.

Theorem 4.2

Let $F \in \mathbb{D}^{1,2}$ be an $\mathcal{F}_T$-measurable random variable. Assume that for a.e. any $t \in [0,T]$ the random variable D_tF belongs to $L^1(Q)$, the process θ belongs to $\mathbb{D}^{1,2}\big(L^2([0,T])\big)$ and $Z(T)F \in \mathbb{D}^{1,2}$. Then

$$F = E_Q(F) + \int_0^T E_Q\left(\left(D_tF - F\int_t^T D_t\theta(s)\, d\tilde{W}_s\right) \,\Big|\, \mathcal{F}_t\right) d\tilde{W}_t. \tag{4.9}$$

Notice that Theorem 4.2 is not a trivial rephrasing of Theorem 4.1 because F is $\mathcal{F}_T$-measurable and not necessarily measurable with respect to the σ-field $\tilde{\mathcal{F}}_T$ generated by the new Wiener process $\tilde{W}$.

Before giving the proof of Theorem 4.2, we need to establish some auxiliary results, as follows.

Lemma 4.4

Consider two probabilities μ and ν on a measurable space $(\Omega, \mathcal{F})$. Assume that $d\nu = f d\mu$, with $f \in L^1(\mu)$. Let X be a random variable defined on $(\Omega, \mathcal{F})$ belonging to $L^1(\nu)$. Let $\mathcal{F}_0 \subset \mathcal{F}$ be a σ-algebra. Then

$$E_\nu(X \mid \mathcal{F}_0) E_\mu(f \mid \mathcal{F}_0) = E_\mu(fX \mid \mathcal{F}_0). \tag{4.10}$$

Proof

Let $B \in \mathcal{F}_0$. By the definition of the conditional expectation,

$$\begin{aligned}\int_B E_\nu(X \mid \mathcal{F}_0) f \, d\mu &= \int_B E_\nu(X \mid \mathcal{F}_0) \, d\nu = \int_B X \, d\nu \\ &= \int_B X f \, d\mu = \int_B E_\mu(fX \mid \mathcal{F}_0) \, d\mu.\end{aligned}$$

Using properties of the conditional expectation, we obtain

$$\begin{aligned}\int_B E_\nu(X \mid \mathcal{F}_0) f \, d\mu &= E_\mu\big(E_\nu(X \mid \mathcal{F}_0) f \mathbf{1}_B\big) \\ &= E_\mu\Big(E_\mu\big(E_\nu(X \mid \mathcal{F}_0) f \mathbf{1}_B \bigm| \mathcal{F}_0\big)\Big) \\ &= E_\mu\big(\mathbf{1}_B E_\nu(X \mid \mathcal{F}_0) E_\mu(f \mid \mathcal{F}_0)\big) \\ &= \int_B E_\nu(X \mid \mathcal{F}_0) E_\mu(f \mid \mathcal{F}_0) \, d\mu.\end{aligned}$$

We conclude by comparing the two results obtained in the preceding computations. □

Applying this lemma to $\mathcal{F}_0 = \mathcal{F}_t$, $\mu = P$, $\nu = Q$, defined in (4.8), $f = Z(T)$, defined in (4.6) and any $G \in L^1(Q)$, we obtain the fundamental identity

$$E_Q(G/\mathcal{F}_t) Z(t) = E\big(Z(T) G / \mathcal{F}_t\big), \tag{4.11}$$

since $Z(t)$ is an $\mathcal{F}_t$-martingale in the space $(\Omega, \mathcal{F}, P)$.

Let $u \in L^2_a$. Assume that $u \in \mathbb{D}^{1,2}\big(L^2[0,T]\big)$. Then,

$$E\left(\int_0^T dt \int_0^T ds \big|D_t u(s)\big|^2\right) = E\left(\int_0^T ds \int_{t \le s} dt \big|D_t u(s)\big|^2\right) < \infty.$$

In particular, for almost every $t \in [0,T]$, the process $\big(D_t u(s), s \in [0,T]\big)$ belongs to L^2_a and $\int_0^T D_t u(s) \, dW_s \in L^2\big(\Omega \times [0,T]\big)$. Hence, Proposition 3.15 yields

$$\begin{aligned}D_t\left(\int_0^T u(s) \, dW_s\right) &= \int_0^T D_t u(s) \, dW_s + u(t), \\ &= \int_t^T D_t u(s) \, dW_s + u(t). \end{aligned} \tag{4.12}$$

Lemma 4.5

Let Q be the probability measure defined in (4.8), Z be the process defined in (4.6). Let $F \in \mathbb{D}^{1,2}$ and θ be a stochastic process satisfying (4.5) and $\theta \in \mathbb{D}^{1,2}\big(L^2[0,T]\big)$. Then the following identity holds:

$$D_t\big(Z(T)F\big) = Z(T)\Big(D_tF - F\Big(\theta(t) + \int_t^T D_t\theta(s)\, d\tilde{W}_s\Big)\Big) \tag{4.13}$$

Proof

By the chain rule,

$$D_t\big(Z(T)F\big) = Z(T)D_tF + FD_t\big(Z(T)\big). \tag{4.14}$$

Using again the chain rule, Remark 4.1 and (4.12), we obtain

$$\begin{aligned} D_t\big(Z(T)\big) &= Z(T)\Big(-D_t\Big(\int_0^T \theta(s)\, dW_s\Big) - \frac{1}{2} D_t\Big(\int_0^T \theta^2(s)\, ds\Big)\Big) \\ &= Z(T)\Big(-\int_t^T D_t\theta(s)\, dW_s - \theta(t) - \int_t^T \theta(s)D_t\big(\theta(s)\big)\, ds\Big) \\ &= Z(T)\Big(-\int_t^T D_t\theta(s)\, d\tilde{W}_s - \theta(t)\Big). \end{aligned} \tag{4.15}$$

Plugging this identity into (4.14) yields the result. □

We can now proceed to the proof of the *generalized Clark-Ocone formula*.

Proof of Theorem 4.2

Set $Y(t) = E_Q(F/\mathcal{F}_t)$. Since F is $\mathcal{F}_T$-measurable, $Y(T) = F$. Let $\Lambda(t) = Z^{-1}(t)$. Notice that

$$\Lambda(t) = \exp\Big(\int_0^t \theta(s)\, d\tilde{W}_s - \frac{1}{2}\int_0^t \theta^2(s)\, ds\Big).$$

Hence $\Lambda(t)$ satisfies the linear equation

$$d\Lambda(t) = \Lambda(t)\theta(t)\, d\tilde{W}_t. \tag{4.16}$$

Applying (4.11) to $G := F$, the standard Clark-Ocone formula given in Theorem 4.1 to $F := E\big(Z(T)F \mid \mathcal{F}_t\big)$ and Lemma 4.2 to $F := E\big(Z(T)F \mid \mathcal{F}_t\big)$ and $F_{\mathcal{G}} := \mathcal{F}_s$ yields

$$\begin{aligned} Y(t) &= \Lambda(t)E\big(Z(T)F \mid \mathcal{F}_t\big) \\ &= \Lambda(t)\Big(E\Big(E\big(Z(T)F \mid \mathcal{F}_t\big)\Big) \\ &\quad + \int_0^T E\Big(D_sE\big(Z(T)F \mid \mathcal{F}_t\big) \,\Big|\, \mathcal{F}_s\Big)\, dW_s\Big) \\ &= \Lambda(t)\Big(E\big(Z(T)F\big) + \int_0^t E\Big(D_s\big(Z(T)F\big) \,\Big|\, \mathcal{F}_s\Big) dW_s\Big) \\ &= \Lambda(t)U(t), \end{aligned}$$

where

$$U(t) = E\big(Z(T)F\big) + \int_0^t E\Big(D_s\big(Z(T)F\big) \,\Big|\, \mathcal{F}_s\Big)\, dW_s.$$

By the Itô formula,

$$\begin{aligned} dY(t) &= U(t)\, d\Lambda(t) + \Lambda(t)\, dU(t) + d\langle U, \Lambda\rangle_t \\ &= Y(t)\theta(t)\, d\tilde{W}_t + \Lambda(t)E\Big(D_t\big(Z(T)F\big) \mid \mathcal{F}_t\Big)\, dW_t \\ &\quad + E\Big(D_t\big(Z(T)F\big) \,\Big|\, \mathcal{F}_t\Big)\Lambda(t)\theta(t)\, dt \\ &= Y(t)\theta(t)\, d\tilde{W}_t + \Lambda(t)E\Big(D_t\big(Z(T)F\big) \,\Big|\, \mathcal{F}_t\Big)\, d\tilde{W}_t. \end{aligned}$$

Indeed, by (4.16)

$$U(t)\, d\Lambda(t) = U(t)\Lambda(t)\theta(t)\, d\tilde{W}_t = Y(t)\theta(t)\, d\tilde{W}_t,$$

and

$$d\langle U, \Lambda\rangle_t = E\Big(D_t\big(Z(T)F\big) \,\Big|\, \mathcal{F}_t\Big)\Lambda(t)\theta(t)\, dt.$$

By (4.13), we get

$$\begin{aligned} dY(t) &= Y(t)\theta(t)\, d\tilde{W}_t \\ &\quad + \Lambda(t)\Big(E\big(Z(T)D_tF \mid \mathcal{F}_t\big) - E\big(Z(T)F\theta(t) \mid \mathcal{F}_t\big) \\ &\quad - E\left(Z(T)F \int_t^T D_t\theta(s) d\tilde{W}_s \,\Big|\, \mathcal{F}_t\right)\Big)\, d\tilde{W}_t. \end{aligned}$$

Applying (4.11) first to $G := D_tF$, then to $G := F\theta(t)$ and $G := F\int_t^T D_t\theta(s)\,d\tilde{W}_s$, we obtain

$$dY(t) = E_Q\left(\left(D_tF - F\int_t^T D_t\theta(s)\,d\tilde{W}(s)\right) \Big| \mathcal{F}_t\right) d\tilde{W}_t.$$

Since $Y(0) = E_Q(F \mid \mathcal{F}_0) = E(F)$, the theorem is formally proved.

The above arguments are correct if the following assumptions are satisfied:

1) $F \in \mathbb{D}^{1,2}$, $F \in L^1(Q)$ and $D_tF \in L^1(Q)$, t-a.e.

2) $\theta \in \mathbb{D}^{1,2}\big(L^2([0,T])\big)$, $F\theta(t)$ and $F\displaystyle\int_t^T D_t\theta(s)\,d\tilde{W}_s$ in $L^1(Q)$, t-a.e.

3) $E\big(Z(T)F \mid \mathcal{F}_t\big) \in \mathbb{D}^{1,2}$.

This can be checked from the assumptions of the theorem. Indeed, the hypothesis $Z(T)F \in \mathbb{D}^{1,2}$ implies $F \in L^1(Q)$ and also $E\big(Z(T)F \mid \mathcal{F}_t\big) \in \mathbb{D}^{1,2}$, because of Lemma 4.2. Moreover, the same hypothesis, together with $\theta \in \mathbb{D}^{1,2}\big(L^2([0,T])\big)$ yield that $F\theta(t)$ and $F\int_t^T D_t\theta(s)\,d\tilde{W}_s$ belong to $L^1(Q)$, t-a.e.

Hence, the theorem is proved. □

4.4 Application to option pricing

Consider two kind of investments, safe and risky. For example, bonds belong to the first type and stocks to the second one. The price dynamics for safe investments is given by

$$dA(t) = \rho(t)A(t)\,dt,$$

where $\rho(t)$ is the interest rate at time t. We suppose $A(t) \neq 0$, t-a.s.

For risky investments, a standard model for the price dynamics is

$$dS(t) = \mu(t)S(t)\,dt + \sigma(t)S(t)\,dW(t),$$

where W is a Wiener process and $\sigma(t) \neq 0$, t-a.s.

We assume that the coefficients in the above equations are adapted stochastic processes satisfying the appropriate conditions ensuring existence and uniqueness of solution.

A ***portfolio*** consists of a random number of assets of each type — safe and risky — and this number varies with time. Let us call them $\xi(t)$, $\eta(t)$, respectively. The portfolio's value at time t is clearly given by

$$V(t) = \xi(t)A(t) + \eta(t)S(t). \tag{4.17}$$

A portfolio is ***self-financing*** if

$$dV(t) = \xi(t)dA(t) + \eta(t)dS(t). \tag{4.18}$$

Notice that by applying the Itô formula to (4.17), we do not obtain (4.18). The condition means that no money is brought in or taken out of the portfolio during the time interval we are considering. One problem in option pricing consists of determining a portfolio $(\xi(t), \eta(t))$ and an initial value $V(0)$ which at a future time T leads to a given value G, called the ***payoff function***. That is

$$V(T) = G, \quad \text{a.s.} \tag{4.19}$$

The form of G depends on the financial model. Later on, we shall give the example of European calls.

From (4.17), we have

$$\xi(t) = \frac{V(t) - \eta(t)S(t)}{A(t)}.$$

Then, using the equation satisfied by $A(t)$ and $S(t)$, we obtain the following stochastic differential equation for the value process V:

$$dV(t) = \Big(\rho(t)V(t) + \big(\mu(t) - \rho(t)\big)\eta(t)S(t)\Big)\,dt + \sigma(t)\eta(t)S(t)\,dW(t). \tag{4.20}$$

Notice that the observed process is in fact $\big(V(t), \eta(t)\big)$ and must be $\mathcal{F}_t$-adapted and satisfy (4.19), (4.20).

This is a problem in *Stochastic Backward Differential Equations* (see for instance ref. [52]). However this theory does not provide explicit solutions. We shall see now how the generalized *Clark-Ocone formula* does the job. We shall not specify all the necessary assumptions to perform rigourously the next computations.

Define

$$\theta(t) = \frac{\mu(t) - \rho(t)}{\sigma(t)},$$

$$\tilde{W}(t) = \int_0^t \theta(s)\,ds + W(t).$$

By *Girsanov's Theorem*, $\tilde{W}$ is a Wiener process with respect to the probability measure Q given in (4.8). We can write equation (4.20) in terms of this new Wiener process as follows:

$$dV(t) = \rho(t)V(t)\,dt + \sigma(t)\eta(t)S(t)\,d\tilde{W}(t). \tag{4.21}$$

Set

$$U(t) = \exp\left(-\int_0^t \rho(s)\,ds\right)V(t). \tag{4.22}$$

Owing to (4.21), we obtain

$$dU(t) = \exp\left(-\int_0^t \rho(s)\,ds\right)\sigma(t)\eta(t)S(t)\,d\tilde{W}(t),$$

or equivalently,

$$\begin{aligned} &\exp\left(-\int_0^T \rho(s)\,ds\right)V(T) \\ &\quad = V(0) + \int_0^T \exp\left(-\int_0^t \rho(s)\,ds\right)\sigma(t)\eta(t)S(t)\,d\tilde{W}(t). \end{aligned} \tag{4.23}$$

Consider the random variable

$$F = \exp\left(-\int_0^T \rho(s)\,ds\right)G \tag{4.24}$$

and apply formula (4.9). We conclude that

$$V(0) = E_Q(F), \tag{4.25}$$

and

$$\begin{aligned} &\exp\left(-\int_0^t \rho(s)\,ds\right)\sigma(t)\eta(t)S(t) \\ &\quad = E_Q\left(\left(D_tF - F\int_t^T D_t\theta(s)d\tilde{W}(s)\,\Big|\,\mathcal{F}_t\right)\right). \end{aligned}$$

Hence,

$$\begin{aligned} \eta(t) = {} & \exp\left(\int_0^t \rho(s)\,ds\right)\sigma^{-1}(t)S^{-1}(t) \\ & \times E_Q\left(\left(D_tF - F\int_t^T D_t\theta(s)\,d\tilde{W}(s)\,\Big|\,\mathcal{F}_t\right)\right). \end{aligned} \tag{4.26}$$

Therefore, the evolution of the number of risky assets can be computed using the generalized Clark-Ocone formula and the characteristics of the market.

Examples 4.1

Consider the particular case where the coefficients ρ, μ, σ do not depend on t and, in addition, $\sigma \neq 0$. Then, since θ is also constant, $D\theta = 0$ and (4.26) becomes

$$\eta(t) = \exp\bigl(\rho(t-T)\bigr)\sigma^{-1}S^{-1}(t)E_Q\bigl(D_tG \mid \mathcal{F}_t\bigr). \tag{4.27}$$

Consider the particular case $G = \exp\bigl(\alpha W(T)\bigr)$, $\alpha \neq 0$. The chain rule of Malliavin calculus yields

$$D_tG = \alpha \exp\bigl(\alpha W(T)\bigr)\,\mathbb{1}_{t\le T} = \alpha \exp\bigl(\alpha W(T)\bigr).$$

Then (4.27) and (4.11) imply that

$$\eta(t) = \alpha \exp\bigl(\rho(t-T)\bigr)\sigma^{-1}S^{-1}(t)Z^{-1}(t)E\Bigl(Z(T)\exp\bigl(\alpha W(T)\bigr)\Bigm| \mathcal{F}_t\Bigr).$$

Consider the martingale $M(t) = \exp\bigl((\alpha-\theta)W(t) - \frac{1}{2}(\alpha-\theta)^2t\bigr)$. Then,

$$Z(T)\exp\bigl(\alpha W(T)\bigr) = M(T)\exp\left(\frac{T}{2}\left((\alpha-\theta)^2-\theta^2\right)\right),$$

and consequently,

$$\eta(t) = \alpha \exp\bigl(\rho(t-T)\bigr)\sigma^{-1}S^{-1}(t)Z^{-1}(t)M(t)\exp\left(\frac{T}{2}\left((\alpha-\theta)^2-\theta^2\right)\right). \tag{4.28}$$

The equation satisfied by the process $S(t)$ is linear with constant coefficients; therefore it can be solved explicitely. Indeed we have

$$S(t) = S(0)\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right). \tag{4.29}$$

Substituting this expresion for $S(t)$ in (4.28), we obtain an explicit value for $\eta(t)$ in terms of α, ρ, μ, σ, θ and $W(t)$, as follows,

$$\begin{aligned}\eta(t) = \alpha\sigma^{-1}S(0)^{-1}\exp\biggl(&\rho(t-T) + (\alpha-\sigma)W(t)\\ &+\frac{T-t}{2}(\alpha-\theta)^2 - \frac{\theta^2T}{2} + \left(\frac{\theta^2+\sigma^2}{2} - \mu\right)t\biggr).\end{aligned}$$

Let us now consider the same situation as in the preceding example for the particular case of stock options called ***European call options***. This type of financial product gives the owner the right (but not the obligation) to buy the risky stock with value $S(T)$ at the maturity time T at a fixed price K.

The strategy of the owner is as follows. If $S(T) > K$ the profit is $S(T) - K$ and he will buy the stock. If $S(T) \le K$ he does not exercise his right and the profit is zero. Hence

$$G = f\bigl(S(T)\bigr),$$

with

$$f(x) = (x - K)^+.$$

Our goal is to apply formula (4.27) for this particular random variable G. Notice that the function f is Lipschitz. Moreover, from (4.29) it clearly follows that $S(T) \in \mathbb{D}^{1,2}$ and $D_t S(T) = \sigma S(T)$. Thus, $G \in \mathbb{D}^{1,2}$ (see Proposition 3.6). Actually DG can be computed by approximating f by means of a sequence of smooth functions (see Proposition 3.5), obtaining

$$D_t G = \mathbf{1}_{[K,\infty)}\big(S(T)\big) S(T)\sigma. \tag{4.30}$$

An alternative argument for checking (4.30) relies on the local property of the operator D. Indeed, on the set $A = \big\{\omega; S(T)(\omega) < K\big\}$, $G = 0$ and on A^c, $G = S(T) - K$.

Thus, by (4.27),

$$\eta(t) = \exp\big(\rho(t-T)\big) S^{-1}(t) E_Q\Big(S(T)\, \mathbf{1}_{[K,\infty)}\big(S(T)\big) \;\Big|\; \mathcal{F}_t\Big).$$

Computations can be made more explicit. Indeed, the process $S(t)$ satisfies $S(0) = y$ and

$$\begin{aligned} dS(t) &= \mu S(t)\, dt + \sigma S(t)\, dW(t) \\ &= \rho S(t)\, dt + \sigma S(t)\, d\tilde{W}(t). \end{aligned}$$

Therefore, $S(t)$ is a diffusion process in the probability space $(\Omega, \mathcal{F}, Q)$ and hence it possesses the Markov property in this space. Moreover,

$$S(t) = S(0) \exp\left(\left(\rho - \frac{\sigma^2}{2}\right) t + \sigma \tilde{W}(t)\right).$$

This yields

$$\begin{aligned} \eta(t) &= \exp\big(\rho(t-T)\big) S^{-1}(t) E_Q^y \Big(S(T-t)\, \mathbf{1}_{[K,\infty)}\big(S(T-t)\big)\Big)\Big|_{y=S(t)} \\ &= \exp\big(\rho(t-T)\big) S^{-1}(t) E^y \Big(Y(T-t)\, \mathbf{1}_{[K,\infty)}\big(Y(T-t)\big)\Big)\Big|_{y=S(t)}, \end{aligned} \tag{4.31}$$

where E_Q^y denotes the conditional expectation E_Q knowing that $S(0) = y$ and

$$Y(t) = S(0) \exp\left(\left(\rho - \frac{\sigma^2}{2}\right) t + \sigma W(t)\right).$$

Since the law of $W(t)$ is known, the value $\eta(t)$ can be written explicitely in terms of quantities involving $S(t)$ and the normal distribution.

Remember that $\eta(t)$ is the number of units of the risky asset we must have in the portfolio at any time $t \le T$ in order to get the payoff $G = \big(S(T) - K\big)^+$, a.s. at time T, and $V(0)$ is the initial capital needed to achieve this goal. Owing to (4.25), (4.24) and the previous computations,

$$\begin{aligned} V(0) &= E_Q(e^{-\rho T} G) = e^{-\rho T} E_Q\big((S(T) - K)^+\big) \\ &= e^{-\rho T} E\Big(\big(Y(T) - K\big)^+\Big), \end{aligned}$$

which, by the same arguments as before, can be computed explicitely.

The expression of $V(0)$ given above is known as the Black-Scholes pricing formula for European call options.

Comments

Malliavin Calculus is currently being applied in the recent field of stochastic financial mathematics. We have chosen here one of these applications —the most well-known— to a problem in option pricing. The choice was made on the basis of its theoretical interest. In fact, it gives further insight to Itô's result on the representation of square integrable random variables. We have followed the lecture notes [51].

The results of Section 4.1 are from Nualart and Zakai (see ref. [47]). Theorem 4.1 has been proved in reference [49] (see also ref. [12]); Theorem 4.2 appears in reference [24].

4.5 Exercises

4.5.1

Let W be a white noise based on $(A, \mathcal{A}, m)$. Consider a random variable $F \in \operatorname{Dom} D^k$ and $G \in \mathcal{A}$. Prove that

$$D^k_{\underline{t}} E(F \mid \mathcal{F}_G) = E(D^k_{\underline{t}} F \mid \mathcal{F}_G)\, \mathbf{1}_{G^k}(\underline{t}).$$

Hint: This is a generalization of Lemma 4.2. Give an expression of the iterated derivative in terms of the Wiener chaos decomposition which generalizes Proposition 3.13.

4.5.2

Using the *Clark-Ocone formula* (4.4), find the integral representation of the following random variables (you can check the result by using Itô's formula):

1) $F = W^2(T)$,

2) $F = W^3(T)$,

3) $F = (W(T) + T)\exp\left(-W(T) - \frac{1}{2}T\right)$.

4.5.3

In the framework of the generalized Clark-Ocone formula (4.9), find the integral representation with respect to the integrator $\tilde{W}$ for the following random variables:

1) $F = W^2(T)$, θ deterministic,

2) $F = \exp\left(\int_0^T \lambda(s)\, dW(s)\right)$, λ, θ deterministic,

3) $F = \exp\left(\int_0^T \lambda(s)\, dW(s)\right)$, λ deterministic and $\theta(s) = W(s)$.

Chapter 5

Criteria for Absolute Continuity and Smoothness of Probability Laws

In Chapter 1, we gave general results ensuring existence and smoothness of the density of probability laws. The assumptions were formulated in terms of the validity of integration by parts formulae or related properties. The purpose now is to analyze under which conditions on the random vectors these assumptions are fullfilled.

The underlying probability space is the one associated with a generic Gaussian process $\big(W(h), h \in H\big)$, as described at the begining of Chapter 3.

5.1 Existence of a density

Let us start with a very simple result whose proof relies on the arguments of Section 2.3.

Proposition 5.1

Let F be a random variable belonging to $\mathbb{D}^{1,2}$. Assume that the random variable $DF/\|DF\|_H^2$ belongs to the domain of δ in $L^2(\Omega)$. Then the law of F is absolutely continuous. Moreover, its density is given by

$$p(x) = E\left(\mathbf{1}_{(F \geq x)}\, \delta\left(\frac{DF}{\|DF\|_H^2}\right)\right) \tag{5.1}$$

and therefore it is continuous and bounded.

Proof

We are going to check that for any $\varphi \in \mathcal{C}_b^\infty(\mathbb{R})$,

$$E\big(\varphi'(F)\big) = E\left(\varphi(F)\,\delta\left(\frac{DF}{\|DF\|_H^2}\right)\right). \tag{5.2}$$

Thus (1.1) holds for $G = 1$ with $H_1(F,1) = \delta\big(\frac{DF}{\|DF\|_H^2}\big)$. Then the results follow from part 1 of Proposition 1.1.

The chain rule of Malliavin calculus yields $D\big(\varphi(F)\big) = \varphi'(F)DF$. Thus,

$$\varphi'(F) = \left\langle D\big(\varphi(F)\big), \frac{DF}{\|DF\|_H^2}\right\rangle_H.$$

Therefore, the integration by parts formula implies

$$\begin{aligned} E\big(\varphi'(F)\big) &= E\left(\left\langle D\big(\varphi(F)\big), \frac{DF}{\|DF\|_H^2}\right\rangle_H\right) \\ &= E\left(\varphi(F)\,\delta\left(\frac{DF}{\|DF\|_H^2}\right)\right), \end{aligned}$$

proving (5.2). □

Remark 5.1 Notice the analogy between (5.2) and the finite dimensional formula (2.16).

If $n > 1$, the analysis is more involved. We illustrate this fact in the next statement. First we introduce a notion that plays a crucial role.

Definition 5.1 Let $F : \Omega \to \mathbb{R}^n$ be a random vector with components $F^j \in \mathbb{D}^{1,2}$, $j = 1, \dots, n$. The ***Malliavin matrix*** of F is the $n \times n$ matrix, denoted by γ, whose entries are the random variables $\gamma_{i,j} = \langle DF^i, DF^j\rangle_H$, $i, j = 1, \dots, n$.

Proposition 5.2

Let $F : \Omega \to \mathbb{R}^n$ be a random vector with components $F^j \in \mathbb{D}^{1,2}$, $j = 1, \dots, n$. Assume that :

1) *The Malliavin matrix γ is inversible, a.s.*
2) *For every $i, j = 1, \dots, n$, the random variables $(\gamma^{-1})_{i,j} DF^j$ belong to* Dom δ.

Then for any function $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$,

$$E\big(\partial_i \varphi(F)\big) = E\big(\varphi(F) H_i(F,1)\big), \tag{5.3}$$

with

$$H_i(F,1) = \sum_{\ell=1}^n \delta\big((\gamma^{-1})_{i,\ell} DF^\ell\big). \tag{5.4}$$

Consequently the law of F *is absolutely continuous.*

PROOF

Fix $\varphi \in \mathcal{C}_b^\infty$. By virtue of the chain rule, we have $\varphi(F) \in \mathbb{D}^{1,2}$ and

$$\begin{aligned}\Big\langle D\big(\varphi(F)\big), DF^\ell\Big\rangle_H &= \sum_{k=1}^n \partial_k \varphi(F) \langle DF^k, DF^\ell\rangle_H \\ &= \sum_{k=1}^n \partial_k \varphi(F) \gamma_{k,\ell},\end{aligned}$$

$\ell = 1, \dots, n$. Since γ is inversible a.s., this system of linear equations in $\partial_k \varphi(F)$, $k = 1, \dots, n$, can be solved. That is,

$$\partial_i \varphi(F) = \sum_{\ell=1}^n \Big\langle D\big(\varphi(F)\big), (\gamma^{-1})_{i,\ell} DF^\ell\Big\rangle_H, \tag{5.5}$$

$i = 1, \dots, n$, a.s.

The assumption 2) and the duality formula yields

$$\begin{aligned}\sum_{\ell=1}^n E\Big(\varphi(F)\, \delta\big((\gamma^{-1})_{i,\ell} DF^\ell\big)\Big) &= \sum_{\ell=1}^n E\Big(\Big\langle D\big(\varphi(F)\big),\, (\gamma^{-1})_{i,\ell} DF^\ell\Big\rangle_H\Big) \\ &= E\big(\partial_i \varphi(F)\big).\end{aligned}$$

Hence (5.3), (5.4) is proved.

Notice that by assumption $H_i(F,1) \in L^2(\Omega)$. Thus part 1 of Proposition 1.2 yields the existence of the density. □

Using Propositions 3.9 and 3.10, one can give sufficient conditions ensuring the validity of the above assumption 2 and an alternative form of the random variables $H_i(F,1)$, as follows.

Corollary 5.1

Assume that the Malliavin matrix γ is inversible a.s. and for any $i,j = 1,\dots,n$, $F^j \in \operatorname{Dom} L$, $(\gamma^{-1})_{i,j} \in \mathbb{D}^{1,2}$, $(\gamma^{-1})_{i,j} DF^j \in L^2(\Omega, H)$, $(\gamma^{-1})_{i,j}\,\delta(DF^j) \in L^2(\Omega)$, $\langle D(\gamma^{-1})_{i,j}, DF^j\rangle_H \in L^2(\Omega)$. Then the conclusion of Proposition 5.2 holds, and moreover

$$H_i(F,1) = -\sum_{j=1}^{n}\Big(\langle DF^j, D(\gamma^{-1})_{i,j})\rangle_H + (\gamma^{-1})_{i,j} LF^j\Big).$$

The assumption of part 2 of Proposition 5.2, as well as the sufficient conditions given in the preceding Corollary are not easy to verify. In the next Proposition we give a statement which is much more suitable for applications.

Theorem 5.1

Let $F : \Omega \longrightarrow \mathbb{R}^n$ be a random vector satisfying the following conditions:

(a) $F^j \in \mathbb{D}^{2,4}$, for any $j = 1,\dots,n$,

(b) the Malliavin matrix is inversible, a.s.

Then the law of F has a density with respect to the Lebesgue measure on $\mathbb{R}^n$.

Proof

As in the proof of Proposition 5.2, we obtain the system of equations (5.5) for any function $\varphi \in \mathcal{C}_b^\infty$. That is,

$$\partial_i \varphi(F) = \sum_{\ell=1}^{n}\Big\langle D\big(\varphi(F)\big), (\gamma^{-1})_{i,\ell} DF^\ell\Big\rangle_H,$$

$i = 1,\dots,n$, a.s.

We would like to take expectations on both sides of this expression. However, assumption (a) does not ensure the integrability of γ^{-1}. We overcome this problem by localising (5.5), as follows.

For any natural number $N \geq 1$, we define the set

$$C_N = \left\{ \sigma \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) : \|\sigma\| \leq N,\ |\det \sigma| \geq \frac{1}{N} \right\}.$$

Then we consider a non-negative function $\psi_N \in \mathcal{C}_0^\infty\big(\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)\big)$ satisfying

(i) $\psi_N(\sigma) = 1$, if $\sigma \in C_N$,

(ii) $\psi_N(\sigma) = 0$, if $\sigma \notin C_{N+1}$.

From (5.5), it follows that

$$E\big(\psi_N(\gamma)\partial_i\varphi(F)\big) = \sum_{\ell=1}^n E\Big(\Big\langle D\big(\varphi(F)\big),\ \psi_N(\gamma) DF^\ell (\gamma^{-1})_{i,\ell} \Big\rangle_H\Big) \tag{5.6}$$

The random variable $\psi_N(\gamma)DF^\ell(\gamma^{-1})_{i,\ell}$ belongs to $\mathbb{D}^{1,2}(H)$, by assumption (a). Consequently $\psi_N(\gamma)DF^\ell(\gamma^{-1})_{i,\ell} \in \operatorname{Dom}\delta$ (see Proposition 3.8). Hence, by the duality identity,

$$\begin{aligned}\Big|E\big(\psi_N(\gamma)\partial_i\varphi(F)\big)\Big| &= \left|\sum_{\ell=1}^n E\Big(\Big\langle D\big(\varphi(F)\big),\ \psi_N(\gamma) DF^\ell (\gamma^{-1})_{i,\ell} \Big\rangle_H\Big)\right| \\ &\leq E\left(\left|\sum_{\ell=1}^n \delta\big(\psi_N(\gamma)DF^\ell(\gamma^{-1})_{i,\ell}\big)\right|\right)\|\varphi\|_\infty.\end{aligned}$$

Proposition 1.2 part 1 (see Remark 1.3) yields the existence of a density for the probability law $P_N \circ F^{-1}$, where P_N denotes the finite measure on $(\Omega, \mathcal{G})$ absolutely continuous with respect to P with density given by $\psi_N(\gamma)$. Therefore, for any $B \in \mathcal{B}(\mathbb{R}^n)$ with Lebesgue measure equal to zero, we have

$$\int_{F^{-1}(B)} \psi_N(\gamma)\, dP = 0.$$

Let $N \to \infty$. Assumption (b) implies that $\lim_{N\to\infty}\psi_N(\gamma) = 1$. Hence, by bounded convergence, we obtain $P\big(F^{-1}(B)\big) = 0$. This finishes the proof of the Proposition. □

Remark 5.2 The assumptions of the above Theorem 5.1 are not optimal. Indeed, Bouleau and Hirsch proved a better result using other techniques in the more general setting of *Dirichlet forms*. For the sake of completenes we give one of their statements, the one most similar to Theorem 5.1, and refer the reader to reference [8] for more complete information.

Proposition 5.3

Let $F : \Omega \to \mathbb{R}^n$ be a random vector satisfying the following conditions:

(a) $F^j \in \mathbb{D}^{1,2}$, for any $j = 1, \ldots, n$,

(b) the Malliavin matrix is inversible, a.s.

Then the law of F has a density with respect to the Lebesgue measure on $\mathbb{R}^n$.

5.2 Smoothness of the density

This section is devoted to the proof of the following result.

Theorem 5.2

Let $F : \Omega \longrightarrow \mathbb{R}^n$ be a random vector satisfying the assumptions

(a) $F^j \in \mathbb{D}^\infty$, for any $j = 1, \ldots, n$,

(b) the Malliavin matrix γ is invertible a.s. and

$$\det \gamma^{-1} \in \bigcap_{p \in [1,\infty)} L^p(\Omega).$$

Then the law of F has an infinitely differentiable density with respect to Lebesgue measure on $\mathbb{R}^n$.

So far we have been dealing with applications of part 1 of Propositions 1.1 and 1.2. That is, we have only considered first order derivatives and the corresponding integration by parts formula (1.1) with $G = 1$.

In order to study the question of smoothness of density, we need iterations of this first-order formula and, in this case, we really need to consider (1.1) for $G \neq 1$. Theorem 5.2 is a consequence of the next proposition and part 2 of Proposition 1.1.

Proposition 5.4

Let $F : \Omega \longrightarrow \mathbb{R}^n$ be a random vector such that $F^j \in \mathbb{D}^\infty$ for any $j = 1, \ldots, n$. Assume that

$$\det \gamma^{-1} \in \bigcap_{p \in [1,\infty)} L^p(\Omega). \tag{5.7}$$

Then:

1) $\det \gamma^{-1} \in \mathbb{D}^\infty$ *and* $\gamma^{-1} \in \mathbb{D}^\infty(\mathbb{R}^m \times \mathbb{R}^m)$.
2) *Let* $G \in \mathbb{D}^\infty$. *For any multiindex* $\alpha \in \{1, \ldots, n\}^r$, $r \geq 1$, *there exists a random variable* $H_\alpha(F, G) \in \mathbb{D}^\infty$ *such that for any function* $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$,

$$E\big((\delta_\alpha \varphi)(F) G\big) = E\big(\varphi(F) H_\alpha(F, G)\big). \tag{5.8}$$

In addition, the random variables $H_\alpha(F, G)$ can be defined recursively as follows: If $|\alpha| = 1$, $\alpha = i$, then

$$H_i(F, G) = \sum_{\ell=1}^n \delta\big(G(\gamma^{-1})_{i,\ell} DF^\ell\big), \tag{5.9}$$

and in general, for $\alpha = (\alpha_1, \ldots, \alpha_{r-1}, \alpha_r)$,

$$H_\alpha(F, G) = H_{\alpha_r}\big(F, H_{(\alpha_1, \ldots, \alpha_{r-1})}(F, G)\big). \tag{5.10}$$

Proof

Consider the sequence of random variables $\big(Y_N = (\det \gamma + \frac{1}{N})^{-1}, N \geq 1\big)$. Assumption (5.7) clearly yields

$$\lim_{N \to \infty} Y_N = \det \gamma^{-1}$$

in $L^p(\Omega)$.

We now prove the following facts:

(a) $Y_N \in \mathbb{D}^\infty$, for any $N \geq 1$,

(b) $(D^k Y_N, n \geq 1)$ is a Cauchy sequence in $L^p(\Omega; H^{\otimes k})$, for any natural number k.

Since the operator D^k is closed, the claim (a) will follow.

Consider the function $\varphi_N(x) = (x + \frac{1}{N})^{-1}$, $x > 0$. Notice that $\varphi_N \in \mathcal{C}_p^\infty$. Then Remark 3.4 yields recursively (a). Indeed, $\det \gamma \in \mathbb{D}^\infty$.

Let us now prove (b). The sequence of derivatives $\big(\varphi_N^{(n)}(\det\gamma), N \geq 1\big)$ is Cauchy in $L^p(\Omega)$, for any $p \in [1,\infty)$. This can be proved using (5.7) and bounded convergence. The result now follows by expressing the difference $D^k Y_N - D^k Y_M$, $N, M \geq 1$, by means of Leibniz's rule (see (3.20)) and using that $\det\gamma \in \mathbb{D}^\infty$.

Once we have proved that $\det\gamma^{-1} \in \mathbb{D}^\infty$, we trivially obtain $\gamma^{-1} \in \mathbb{D}^\infty(\mathbb{R}^m \times \mathbb{R}^m)$, by a direct computation of the inverse of a matrix and using that $F^j \in \mathbb{D}^\infty$. The proof of (5.10) is done by induction on the order r of the multiindex α. Let $r = 1$. Consider the identity (5.5), multiply both sides by G and take expectations. We obtain (5.8) and (5.9).

Assume that (5.8) holds for multiindices of order $k-1$. Fix $\alpha = (\alpha_1, \dots, \alpha_{k-1}, \alpha_k)$. Then,

$$\begin{aligned} E\big((\partial_\alpha \varphi)(F)G\big) &= E\Big(\partial_{(\alpha_1,\dots,\alpha_{k-1})}\big((\partial_{\alpha_k}\varphi)(F)\big)G\Big) \\ &= E\big((\partial_{\alpha_k}\varphi)(F) H_{(\alpha_1,\dots,\alpha_{k-1})}(F,G)\big) \\ &= E\Big(\varphi(F) H_{\alpha_k}\big(F, H_{(\alpha_1,\dots,\alpha_{k-1})}(F,G)\big)\Big). \end{aligned}$$

The proof is complete. □

COMMENTS

The results of this chapter are either rephrasings or quotations of statements from references [8], [19], [43], [48], [63] and [68], just to mention a few of them. The common source is reference [33].

Chapter 6

Stochastic Partial Differential Equations Driven by Spatially Homogeneous Gaussian Noise

The purpose of the rest of this book is to apply the criteria established in Chapter 5 to random vectors which are solutions of SPDE's. As preliminaries, we give in this chapter a result on existence and uniqueness of solutions for a general type of equations that cover all the cases we shall encounter. We start by introducing the stochastic integral to be used in the rigourous formulation of the SPDE's and in the application of the Malliavin differential calculus. This is an extension of the integral studied in reference [14] and developed in reference [54].

6.1 Stochastic integration with respect to coloured noise

Throughout this section, we will use the notations and notions concerning distributions given in reference [61]. Let $\mathcal{D}(\mathbb{R}^{d+1})$ be the space of *Schwartz test functions* and $\mathcal{S}(\mathbb{R}^{d+1})$ be the set of $\mathcal{C}^\infty$ *functions with rapid decrease*. We recall that $\mathcal{D}(\mathbb{R}^{d+1}) \subset \mathcal{S}(\mathbb{R}^{d+1})$. Generic elements of this space shall be denoted by $\varphi(s,x)$. For any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ we define the *Fourier transform* as

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\langle x,\xi\rangle}\varphi(x)\,dx, \qquad \xi \in \mathbb{R}^d,$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product in $\mathbb{R}^d$. Let Γ be a non-negative, non-negative definite *tempered measure*. Define

$$J(\varphi,\psi) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \Gamma(dx)\big(\varphi(s) * \tilde{\psi}(s)\big)(x), \tag{6.1}$$

where $\tilde{\psi}(s,x) = \psi(s,-x)$ and the symbol "$*$" denotes the *convolution* operation. According to reference [61] (Chap. VII, Theoreme XVII), the measure Γ is symmetric. Hence the functional J defines an inner product on $\mathcal{D}(\mathbb{R}^{d+1}) \times \mathcal{D}(\mathbb{R}^{d+1})$. Moreover, there exists a non-negative *tempered measure* μ on $\mathbb{R}^d$ whose Fourier transform is Γ (see Théorème XVIII of chap. VII of ref. [61]). Therefore,

$$J(\varphi,\psi) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \mu(d\xi)\mathcal{F}\varphi(s)(\xi)\overline{\mathcal{F}\psi(s)(\xi)}. \tag{6.2}$$

There is a natural Hilbert space associated with the *covariance functional* J. Indeed, let $\mathcal{E}$ be the inner product space consisting of functions $\varphi \in \mathcal{S}(\mathbb{R}^d)$, endowed with the inner product

$$\langle \varphi,\psi\rangle_{\mathcal{E}} := \int_{\mathbb{R}^d} \Gamma(dx)(\varphi * \tilde{\psi})(x),$$

where $\tilde{\psi}(x) = \psi(-x)$. Notice that

$$\langle \varphi,\psi\rangle_{\mathcal{E}} = \int_{\mathbb{R}^d} \mu(d\xi)\mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}. \tag{6.3}$$

Let $\mathcal{H}$ denote the completion of $\big(\mathcal{E}, \langle\cdot,\cdot\rangle_{\mathcal{E}}\big)$. Set $\mathcal{H}_T = L^2\big([0,T];\mathcal{H}\big)$. The inner product in $\mathcal{H}_T$ extends that defined in (6.1). On a fixed probability space $(\Omega,\mathcal{G},P)$, we consider a zero mean Gaussian stochastic process $F = \big(F(\varphi), \varphi \in \mathcal{D}(\mathbb{R}^{d+1})\big)$, with covariance functional given by (6.1). We shall derive from F a stochastic process

$$M = \big(M_t(A),\, t \ge 0,\, A \in \mathcal{B}_b(\mathbb{R}^d)\big)$$

which shall act as integrator.

Fix a rectangle R in $\mathbb{R}^{d+1}$. Let $(\varphi_n, n \ge 0) \subset \mathcal{D}(\mathbb{R}^{d+1})$ be such that $\lim_{n\to\infty} \varphi_n = \mathbf{1}_R$ pointwise. Then, by bounded convergence it follows that

$$\lim_{n,m\to\infty} E\big(F(\varphi_n) - F(\varphi_m)\big)^2 = 0.$$

Set $F(R) = \lim_{n\to\infty} F(\varphi_n)$, in $L^2(\Omega)$. It is easy to check that the limit does not depend on the particular approximating sequence. This extension of F trivially holds for finite unions of rectangles. If R^1, R^2 are two such elements one proves, using again bounded convergence, that

$$E\big(F(R^1)F(R^2)\big) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \Gamma(dx)\big(\mathbf{1}_{R^1}(s) * \tilde{\mathbf{1}}_{R^2}(s)\big)(x).$$

In addition, if $R_n, n \geq 0$, is a sequence of finite unions of rectangles decreasing to $\emptyset$, then the same kind of arguments yield $\lim_{n\to\infty} E\big(F(R_n)^2\big) = 0$. Hence the mapping $R \to F(R)$ can be extended to an $L^2(P)$-valued measure defined on $\mathcal{B}_b(\mathbb{R}^{d+1})$.

For any $t \geq 0$, $A \in \mathcal{B}_b(\mathbb{R}^d)$, set $M_t(A) = F\big([0,t] \times A\big)$. Let $\mathcal{G}_t$ be the completion of the σ-field generated by the random variables $M_s(A)$, $0 \leq s \leq t$, $A \in \mathcal{B}_b(\mathbb{R}^d)$. The properties of F ensure that the process

$$M = \big(M_t(A), t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)\big),$$

is a martingale with respect to the filtration $\big(\mathcal{G}_t, t \geq 0\big)$. Thus the process M is a ***martingale measure*** (see ref. [67], p. 287).

The covariance functional coincides with the mutual variation; it is given by

$$\big\langle M(A), M(B)\big\rangle_t = t \int_{\mathbb{R}^d} \Gamma(dx)(\mathbf{1}_A * \tilde{\mathbf{1}}_B)(x).$$

The *dominating measure* (see ref. [67], p. 291) coincides with the covariance functional.

The theory of stochastic integration with respect to martingale measures developed by Walsh allows to integrate predictable stochastic processes $\big(X(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\big)$ satisfying the integrability condition

$$E\left(\int_0^T ds \int_{\mathbb{R}^d} \Gamma(dx)\big(|X|(s) * |\tilde{X}|(s)\big)(x)\right) < \infty.$$

In the context of SPDE's this integral is not always appropriate. Consider for instance the *stochastic wave equation* in dimension $d \geq 3$. The fundamental solution is a distribution. Therefore in evolution formulations of the equation we shall meet integrands which include a deterministic distribution-valued function. With this problem as motivation, Dalang has extended in reference [14] Walsh's stochastic integral. In the remainder of this section, we shall review his ideas in the more general context of Hilbert-valued integrands. This extension is needed when dealing with the Malliavin derivatives of the solutions of SPDE's.

Let $\mathcal{K}$ be a separable real Hilbert space with inner product and norm denoted by $\langle\cdot,\cdot\rangle_{\mathcal{K}}$ and $\|\cdot\|_{\mathcal{K}}$, respectively. Let $K = \big\{K(s,z), (s,z) \in [0,T] \times \mathbb{R}^d\big\}$ be a $\mathcal{K}$-valued predictable process; we assume the following condition:

Hypothesis B. *The process K satisfies*

$$\sup_{(s,z)\in[0,T]\times\mathbb{R}^d} E\Big(\big\|K(s,z)\big\|_{\mathcal{K}}^2\Big) < \infty.$$

Our first purpose is to define a martingale measure with values in $\mathcal{K}$ obtained by integration of K. Let $\{e_j, j \geq 0\}$ be a *orthonormal system* of $\mathcal{K}$. Set $K^j(s,z) = \langle K(s,z), e_j \rangle_{\mathcal{K}}$, $(s,z) \in [0,T] \times \mathbb{R}^d$. According to reference [67], for any $j \geq 0$ the process

$$M_t^{K^j}(A) = \int_0^t \int_A K^j(s,z) M(ds,dz), \qquad t \in [0,T],\ A \in \mathcal{B}_b(\mathbb{R}^d),$$

defines a martingale measure. Indeed, the process K^j is predictable and

$$\sup_{(s,z)\in[0,T]\times\mathbb{R}^d} E\Big(\big|K^j(s,z)\big|^2\Big) \leq \sup_{(s,z)\in[0,T]\times\mathbb{R}^d} E\Big(\big\|K(s,z)\big\|_{\mathcal{K}}^2\Big) < \infty,$$

which yields

$$E\left(\int_0^t ds \int_{\mathbb{R}^d} \Gamma(dx) \big(\mathbf{1}_A K^j(s) * \tilde{\mathbf{1}}_A \tilde{K}^j(s)\big)(x)\right) < \infty.$$

Set, for any $t \in [0,T]$, $A \in \mathcal{B}_b(\mathbb{R}^d)$,

$$M_t^K(A) = \sum_{j\geq 0} M_t^{K^j}(A) e_j. \tag{6.4}$$

The right hand-side of (6.4) defines an element of $L^2(\Omega;\mathcal{K})$. Indeed, using the *isometry property* of the stochastic integral, *Parseval's identity* and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\sum_{j\geq 0} E\Big(\big|M_t^{K^j}(A)\big|^2\Big) \\
&= \sum_{j\geq 0} E\left(\left|\int_0^t \int_{\mathbb{R}^d} \mathbf{1}_A(z) K^j(s,z) M(ds,dz)\right|^2\right) \\
&= \sum_{j\geq 0} E\left(\int_0^t ds \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy\, \mathbf{1}_A(y) K^j(s,y)\, \mathbf{1}_A(y-x) K^j(s,y-x)\right) \\
&= \int_0^t ds \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy\, \mathbf{1}_A(y)\, \mathbf{1}_A(y-x) E\Big(\big\langle K(s,y), K(s,y-x)\big\rangle_{\mathcal{K}}\Big) \\
&\leq \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(\big\|K(t,x)\big\|_{\mathcal{K}}^2\Big) \int_0^t ds \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy\, \mathbf{1}_A(y)\, \mathbf{1}_A(y-x) \\
&\leq C \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(\big\|K(t,x)\big\|_{\mathcal{K}}^2\Big).
\end{aligned}$$

This shows that

$$E\Big(\big\|M_t^K(A)\big\|_{\mathcal{K}}^2\Big) = \sum_{j\geq 0} E\Big(\big|M_t^{K^j}(A)\big|^2\Big) < \infty,$$

due to Hypothesis B.

Clearly, the process $\{M_t^K(A),\ t\in[0,T],\ A\in\mathcal{B}_b(\mathbb{R}^d)\}$ defines a worthy $\mathcal{K}$-valued martingale measure, and by construction we have that $\langle M_t^K(A), e_j\rangle_{\mathcal{K}} = M_t^{K^j}(A)$.

By the previous computations

$$E\Big(\big\|M_t^K(A)\big\|_{\mathcal{K}}^2\Big) = \sum_{j\geq 0} E\left(\int_0^t ds \big\|\mathbf{1}_A(\cdot)K^j(s,\cdot)\big\|_{\mathcal{H}}^2\right),$$

where we have denoted by "$\cdot$" the $\mathcal{H}$-variable.

Our next goal is to introduce stochastic integration with respect to M^K, allowing the integrands to take values on some subset of the space of Schwartz distributions. First we briefly recall Walsh's construction in a Hilbert-valued context.

A stochastic process $\{g(s,z;\omega), (s,z)\in[0,T]\times\mathbb{R}^d\}$ is called *elementary* if

$$g(s,z;\omega) = \mathbf{1}_{(a,b]}(s)\,\mathbf{1}_A(z)X(\omega),$$

for some $0\leq a<b\leq T$, $A\in\mathcal{B}_b(\mathbb{R}^d)$ and X a bounded $\mathcal{F}_a$-measurable random variable. For such g the stochastic integral $g\cdot M^K$ is the $\mathcal{K}$-valued martingale measure defined by

$$\big(g\cdot M^K\big)_t(B)(\omega) = \big(M_{t\wedge b}^K(A\cap B) - M_{t\wedge a}^K(A\cap B)\big)X(\omega),$$

$t\in[0,T]$, $B\in\mathcal{B}_b(\mathbb{R}^d)$.

This definition is extended by linearity to the set $\mathcal{E}_s$ of all linear combinations of elementary processes.

For $g\in\mathcal{E}_s$ and $t\geq 0$ one easily checks that

$$\begin{aligned}
&E\Big(\big\|(g\cdot M^K)_t(B)\big\|_{\mathcal{K}}^2\Big)\\
&\quad= \sum_{j\geq 0} E\left(\int_0^t ds\int_{\mathbb{R}^d}\Gamma(dx)\int_{\mathbb{R}^d} dy\, 1_B(y)g(s,y)K^j(s,y)1_B(y-x)\right.\\
&\qquad\left.\times\ g(s,y-x)K^j(s,y-x)\right)\\
&\quad\leq \|g\|_{+,K}^2,
\end{aligned}\tag{6.5}$$

where

$$\|g\|_{+,K}^2 := \sum_{j\geq 0} E\left(\int_0^T ds\Big\|\big|g(s,\cdot)K^j(s,\cdot)\big|\Big\|_{\mathcal{H}}^2\right).$$

Let $\mathcal{P}_{+,K}$ be the set of all predictable processes g such that $\|g\|_{+,K}<\infty$. Then, owing to Exercise 2.5 and Proposition 2.3 of reference [67], $\mathcal{P}_{+,K}$

is complete and $\mathcal{E}_s$ is dense in this Banach space. Thus, we use the bound (6.5) to define the stochastic integral $g \cdot M^K$ for $g \in \mathcal{P}_{+,K}$.

Next, following reference [14] we aim to extend the above stochastic integral to include a larger class of integrands.

Consider the inner product defined on $\mathcal{E}_s$ by the formula

$$\langle g_1, g_2 \rangle_{0,K} = \sum_{j \geq 0} E\left(\int_0^T ds \langle g_1(s,\cdot)K^j(s,\cdot), g_2(s,\cdot)K^j(s,\cdot) \rangle_{\mathcal{H}} \right)$$

and the norm $\|\cdot\|_{0,K}$ derived from it. By the first equality in (6.5) we have that

$$E\Big(\big\| (g \cdot M^K)_T(\mathbb{R}^d) \big\|_{\mathcal{A}}^2 \Big) = \|g\|_{0,K}^2$$

for any $g \in \mathcal{E}_s$.

Let $\mathcal{P}_{0,K}$ be the completion of the inner product space $\big(\mathcal{E}_s, \langle \cdot, \cdot \rangle_{0,K}\big)$. Since $\|\cdot\|_{0,K} \leq \|\cdot\|_{+,K}$, the space $\mathcal{P}_{0,K}$ will be in general larger than $\mathcal{P}_{+,K}$. So, we can extend the stochastic integral with respect to M^K to elements of $\mathcal{P}_{0,K}$. Let $\big(\mathcal{M}, \|\cdot\|_{\mathcal{M}}\big)$ be the space of $\mathcal{K}$-valued continuous square integrable martingales endowed with the norm $\|X\|_{\mathcal{M}}^2 = E\big(\|X_T\|_{\mathcal{K}}^2\big)$. Then the map $g \mapsto g \cdot M^K$, where $g \cdot M^K$ denotes the martingale $t \mapsto (g \cdot M^K)_t(\mathbb{R}^d)$, is an isometry between the spaces $\big(\mathcal{P}_{0,K}, \|\cdot\|_{0,K}\big)$ and $\big(\mathcal{M}, \|\cdot\|_{\mathcal{M}}\big)$. Here we still have denoted by $\|\cdot\|_{0,K}$ the norm derived from the inner product of the completion of $\big(\mathcal{E}_s, \langle \cdot, \cdot \rangle_{0,K}\big)$. Classical results on Hilbert spaces tell us precisely how this norm is constructed (see for instance § 2 of Chap. V of ref. [7]).

In the sequel we denote either by

$$\big(g \cdot M^K\big)_t \quad \text{or} \quad \int_0^t \int_{\mathbb{R}^d} g(s,z)K(s,z)M(ds,dz)$$

the martingale obtained by stochastic integration of $g \in \mathcal{P}_{0,K}$ with respect to M^K.

Let us consider the particular case where the following stationary assumption is fulfilled.

Hypothesis C. *For all $j \geq 0$, $s \in [0,T]$, $x, y \in \mathbb{R}^d$,*

$$E\big(K^j(s,x)K^j(s,y)\big) = E\big(K^j(s,0)K^j(s,y-x)\big).$$

Consider the *non-negative definite function*

$$G_j^K(s,z) := E\big(K^j(s,0)K^j(s,z)\big).$$

Owing to Theorem XIX of Chapter VII of reference [61], the measure $\Gamma^K_{j,s}(dz) = G^K_j(s,z) \times \Gamma(dz)$, is a *non-negative definite distribution*. Thus, by *Bochner's theorem* (see for instance Theorem XVIIII of Chap. VII of ref. [61]) there exists a non-negative tempered measure $\mu^K_{j,s}$ such that $\Gamma^K_{j,s}(dz) = \mathcal{F}\mu^K_{j,s}$.

Clearly, the measure $\Gamma^K_s(dz) := \sum_{j\geq 0} \Gamma^K_{j,s}(dz)$ is a well defined non-negative definite measure on $\mathbb{R}^d$, because

$$\sum_{j\geq 0} G^K_j(s,z) \leq \sup_{(s,z)\in[0,T]\in\mathbb{R}^d} E\Big(\|K(s,z)\|^2_{\mathcal{K}}\Big) < \infty.$$

Consequently, there exists a non-negative tempered measure μ^K_s such that $\mathcal{F}\mu^K_s = \Gamma^K_s$. Furthermore, by the uniqueness and linearity of the Fourier transform, $\mu^K_s = \sum_{j\geq 0} \mu^K_{j,s}$.

Thus, if Hypotheses B and C are satisfied then, for any *deterministic* function $g(s,z)$ such that $\|g\|^2_{0,K} < \infty$ and $g(s) \in \mathcal{S}(\mathbb{R}^d)$ we have that

$$\begin{aligned}
\|g\|^2_{0,K} &= \sum_{j\geq 0} \int_0^T ds \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy\, g(s,y) g(s,y-x) G^K_j(s,x) \\
&= \int_0^T ds \int_{\mathbb{R}^d} \Gamma^K_s(dx) \big(g(s,\cdot) * \tilde{g}(s,\cdot)\big)(x) \\
&= \int_0^T ds \int_{\mathbb{R}^d} \mu^K_s(d\xi) \big|\mathcal{F}g(s)(\xi)\big|^2. \qquad (6.6)
\end{aligned}$$

We now want to give examples of deterministic distribution-valued functions $t \to S(t)$ belonging to $\mathcal{P}_{0,K}$. A result in this direction is given in the next theorem, which is the Hilbert-valued counterpart of Theorems 2 and 5 of reference [14].

Theorem 6.1

Let $\{K(s,z), (s,z) \in [0,T] \times \mathbb{R}^d\}$ be a $\mathcal{K}$-valued process satisfying Hypothesis B and C. Let $t \mapsto S(t)$ be a deterministic function with values in the space of non-negative distributions with rapid decrease, such that

$$\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi) \big|\mathcal{F}S(t)(\xi)\big|^2 < \infty.$$

Then S belongs to $\mathcal{P}_{0,K}$ and

$$E\Big(\|(S\cdot M^K)_t\|^2_{\mathcal{K}}\Big) = \int_0^t ds \int_{\mathbb{R}^d} \mu^K_s(d\xi) \big|\mathcal{F}S(s)(\xi)\big|^2. \qquad (6.7)$$

Moreover, for any $p \in [2,\infty)$,

$$
\begin{aligned}
&E\Big(\big\|(S\cdot M^K)_t\big\|_{\mathcal{K}}^p\Big)\\
&\qquad \le C_t \int_0^t ds \sup_{x\in\mathbb{R}^d} E\Big(\big\|K(s,x)\big\|_{\mathcal{K}}^p\Big)\int_{\mathbb{R}^d}\mu(d\xi)\big|\mathcal{F}S(s)(\xi)\big|^2, \qquad (6.8)
\end{aligned}
$$

with

$$
C_t = \left(\int_0^t\int_{\mathbb{R}^d}\mu(d\xi)\big|\mathcal{F}S(s)(\xi)\big|^2\right)^{\frac{p}{2}-1}, \qquad t\in[0,T].
$$

Proof

Let ψ be a non-negative function in $\mathcal{C}^\infty(\mathbb{R}^d)$ with support contained in the unit ball of $\mathbb{R}^d$ and such that $\int_{\mathbb{R}^d}\psi(x)\,dx = 1$. Set $\psi_n(x) = n^d\psi(nx)$, $n\ge 1$. Define $S_n(t) = \psi_n * S(t)$. Clearly, $S_n(t)\in\mathcal{S}(\mathbb{R}^d)$ for any $n\ge 1$, $t\in[0,T]$ and $S_n(t)\ge 0$.

We first prove that $S_n\in\mathcal{P}_{+,K}\subset\mathcal{P}_{0,K}$. The definition of the norm $\|\cdot\|_{+,K}$, Parseval's identity and Schwarz's inequality yield

$$
\begin{aligned}
\big\|S_n\big\|_{+,K}^2 &= \sum_{j\ge 0} E\left(\int_0^T ds\Big\|\big|S_n(s,\cdot)K^j(s,\cdot)\big|\Big\|_{\mathcal{H}}^2\right)\\
&= \sum_{j\ge 0} E\left(\int_0^T ds\int_{\mathbb{R}^d}\Gamma(dx)\right.\\
&\qquad \left.\times\int_{\mathbb{R}^d} dy\, S_n(s,y)K^j(s,y)S_n(s,y-x)K^j(s,y-x)\right)\\
&= \int_0^T ds\int_{\mathbb{R}^d}\Gamma(dx)\\
&\qquad \times\int_{\mathbb{R}^d} dy\, S_n(s,y)S_n(s,y-x)E\Big(\big\langle K(s,y),K(s,y-x)\big\rangle_{\mathcal{K}}\Big)\\
&\le \int_0^T ds\sup_{x\in\mathbb{R}^d} E\Big(\big\|K(s,x)\big\|_{\mathcal{K}}^2\Big)\int_{\mathbb{R}^d}\mu(d\xi)\big|\mathcal{F}S_n(t)(\xi)\big|^2.
\end{aligned}
$$

Since $\sup_n\big|\mathcal{F}S_n(t)(\xi)\big| \le \big|\mathcal{F}S(t)(\xi)\big|$, this implies

$$
\sup_n\big\|S_n\big\|_{+,K} < \infty. \qquad (6.9)
$$

Let us now show that

$$
\lim_{n\to\infty}\big\|S_n - S\big\|_{0,K} = 0. \qquad (6.10)
$$

We have

$$\begin{aligned}\|S_n - S\|_{0,K}^2 &= \int_0^T dt \int_{\mathbb{R}^d} \mu_t^K(d\xi) \Big|\mathcal{F}\big(S_n(t) - S(t)\big)\Big|^2\\ &= \int_0^T dt \int_{\mathbb{R}^d} \mu_t^K(d\xi) \big|\mathcal{F}\psi_n(\xi) - 1\big|^2 \big|\mathcal{F}S(t)(\xi)\big|^2.\end{aligned}$$

The properties of ψ clearly yield $\big|\mathcal{F}\psi_n(\xi)\big| \le 1$. Moreover, $\mathcal{F}\psi_n(\xi)$ converges pointwise to 1 as n tends to infinity. Then, since $\big|\mathcal{F}\psi_n(\xi) - 1\big| \le 2$, by bounded convergence it suffices to check that

$$\|S\|_{0,K}^2 = \int_0^T dt \int_{\mathbb{R}^d} \mu_t^K(d\xi) \big|\mathcal{F}S(t)(\xi)\big|^2 < \infty.$$

We know that $\big|\mathcal{F}S_n(t)(\xi)\big|$ converges pointwise to $\big|\mathcal{F}S(t)(\xi)\big|$ and

$$\|S_n\|_{0,K}^2 = \int_0^T dt \int_{\mathbb{R}^d} \mu_t^K(d\xi) \big|\mathcal{F}S_n(t)\big|^2.$$

Then, *Fatou's lemma* implies

$$\|S\|_{0,K}^2 \le \liminf_{n\to\infty} \|S_n\|_{0,K}^2 \le \liminf_{n\to\infty} \|S_n\|_{+,K}^2 < \infty,$$

by (6.9). This finish the proof of (6.10) and therefore $S \in \mathcal{P}_{0,K}$.

By the isometry property of the stochastic integral and (6.6) we see that the equality (6.7) holds for any S_n; then the construction of the stochastic integral yields

$$\begin{aligned}E\Big(\Big\|\big(S \cdot M^K\big)_t\Big\|_{\mathcal{K}}^2\Big) &= \lim_{n\to\infty} E\Big(\Big\|\big(S_n \cdot M^K\big)_t\Big\|_{\mathcal{K}}^2\Big)\\ &= \lim_{n\to\infty} \int_0^t ds \int_{\mathbb{R}^d} \mu_s^K(d\xi) \Big|\mathcal{F}\big(S_n(s)(\xi)\big)\Big|^2\\ &= \int_0^t ds \int_{\mathbb{R}^d} \mu_s^K(d\xi) \Big|\mathcal{F}\big(S(s)(\xi)\big)\Big|^2,\end{aligned}$$

where the last equality follows from bounded convergence. This proves (6.7).

We now prove (6.8). The previous computations yield

$$\lim_{k\to\infty} \Big\|\big(S_{n_k} \cdot M^K\big)_t\Big\|_{\mathcal{K}} = \Big\|\big(S \cdot M^K\big)_t\Big\|_{\mathcal{K}}, \qquad (6.11)$$

a.s. for some subsequence n_k, $k \ge 1$. By Fatou's Lemma,

$$E\Big(\Big\|\big(S \cdot M^K\big)_t\Big\|_{\mathcal{K}}^p\Big) \le \liminf_{k\to\infty} E\Big(\Big\|\big(S_{n_k} \cdot M^K\big)_t\Big\|_{\mathcal{K}}^p\Big).$$

In the sequel we shall write S_n instead of S_{n_k}, for the sake of simplicity. Since each S_n is smooth, the stochastic integral $S_n \cdot M^K$ is a classical one (in Walsh's sense). The stochastic process $\big((S_{n_k} \cdot M^K)_t, t \geq 0\big)$ is a $\mathcal{K}$-valued martingale. Then, *Burkholder's inequality* for Hilbert-valued martingales (see ref. [36]) and Schwarz's inequality ensure that

$$\begin{aligned}
&E\Big(\big\|(S_n \cdot M^K)_t\big\|_{\mathcal{K}}^p\Big) \\
&\quad\leq CE\Bigg(\sum_{j\geq 0}\int_0^t ds \int_{\mathbb{R}^d}\Gamma(dx) \\
&\qquad\times \int_{\mathbb{R}^d} dy\, S_n(s,y)S_n(s,x-y)K^j(s,y)K^j(s,x-y)\Bigg)^{\frac{p}{2}} \\
&\quad\leq CE\Bigg(\int_0^t ds \int_{\mathbb{R}^d}\Gamma(dx) \\
&\qquad\times \int_{\mathbb{R}^d} dy\, S_n(s,y)S_n(s,x-y)\big\|K(s,y)\big\|_{\mathcal{K}}\big\|K(s,x-y)\big\|_{\mathcal{K}}\Bigg)^{\frac{p}{2}}.
\end{aligned}$$

For each $n \geq 1$, $t \in [0,T]$, the measure on $[0,t] \times \mathbb{R}^d \times \mathbb{R}^d$ given by $S_n(s,y)S_n(s,x-y)\,ds\,\Gamma(dx)\,dy$ is finite. Indeed,

$$\begin{aligned}
\sup_{n,t}\int_0^t ds \int_{\mathbb{R}^d}\Gamma(dx)\int_{\mathbb{R}^d} dy\, S_n(s,y)S_n(s,x-y)& \\
\leq \sup_{n,t}\int_0^t ds \int_{\mathbb{R}^d}\mu(d\xi)\big|\mathcal{F}S_n(s)(\xi)\big|^2& \\
\leq \int_0^T ds \int_{\mathbb{R}^d}\mu(d\xi)\big|\mathcal{F}S(s)(\xi)\big|^2&.
\end{aligned}$$

Thus, Hölder's inequality applied to this measure yields that the last term in (6.12) is bounded by

$$\begin{aligned}
&C\Bigg(\int_0^T ds \int_{\mathbb{R}^d}\mu(d\xi)\big|\mathcal{F}S(s)(\xi)\big|^2\Bigg)^{\frac{p}{2}-1}\int_0^t ds\int_{\mathbb{R}^d}\Gamma(dx) \\
&\qquad\times \int_{\mathbb{R}^d} dy\, S_n(s,y)S_n(s,x-y)E\Big(\big\|K(s,y)\big\|_{\mathcal{K}}^{\frac{p}{2}}\,\big\|K(s,x-y)\big\|_{\mathcal{K}}^{\frac{p}{2}}\Big).
\end{aligned}$$

Finally, using Hypothesis B one gets,

$$\begin{aligned}
E\Big(\big\|(S \cdot M^K)_t\big\|_{\mathcal{K}}^p\Big) &\leq C\Bigg(\int_0^T ds \int_{\mathbb{R}^d}\mu(d\xi)\big|\mathcal{F}S(s)(\xi)\big|^2\Bigg)^{\frac{p}{2}-1} \\
&\qquad\times \int_0^t ds \sup_{x\in\mathbb{R}^d} E\Big(\big\|K(s,x)\big\|_{\mathcal{K}}^p\Big)\int_{\mathbb{R}^d}\mu(d\xi)\big|\mathcal{F}S_n(s)(\xi)\big|^2.
\end{aligned}$$

Therefore (6.8) holds true. □

Remark 6.1 From the identity (6.7) it follows that for any S satisfying the assumptions of Theorem 6.1 we have

$$\|S\|_{0,K}^2 = \int_0^T ds \int_{\mathbb{R}^d} \mu_s^K(d\xi) \big|\mathcal{F}S(s)(\xi)\big|^2.$$

Remember that we shall also use the notation

$$\int_0^t \int_{\mathbb{R}^d} S(s,y)K(s,y)M(ds,dy),$$

for the stochastic integral of Theorem 6.1.

Remark 6.2 For the sake of completeness we stress that if $\mathcal{K} = \mathbb{R}$, then the assumptions B and C on the real-valued process K read

$$\sup_{(s,x)\in[0,T]\times\mathbb{R}^d} E\Big(\big|K(s,x)\big|^2\Big) < \infty,$$
$$E\big(K(s,x)K(s,y)\big) = E\big(K(s,0)K(s,y-x)\big),$$

respectively.

6.2 Stochastic partial differential equations driven by coloured noise

We are interested in the study of initial-value stochastic partial differential equations driven by Gaussian noises which are white in time and correlated in space. The abstract setting is

$$Lu(t,x) = \sigma\big(u(t,x)\big)\dot{F}(t,x) + b\big(u(t,x)\big), \tag{6.13}$$

$t \in [0,T]$, $x \in \mathbb{R}^d$.

L is a *differential operator*, the coefficients σ and b are real-valued globally Lipschitz functions and $\dot{F}$ is the *formal* differential of the Gaussian process introduced in the previous section.

We must specify the *initial conditions*. For example, if L is of parabolic type (for example, the *heat operator*) we impose

$$u(0,x) = u_0(x).$$

If L is hyperbolic (for instance, the *d'Alembertian operator*) we fix

$$u(0,x) = u_0(x), \qquad \partial_t u(t,x)\big|_{t=0} = v_0(x).$$

We shall assume that the initial conditions vanish. This allows an unified approach for parabolic and hyperbolic operators.

Let us formulate the assumptions concerning the differential operator L and the correlation of the noise.

Hypothesis D. *The fundamental solution Λ of $Lu = 0$ is a deterministic function in t taking values in the space of non-negative* ***measures with rapid decrease*** *(as a distribution). Moreover,* $\sup_{t\in[0,T]} \Lambda(t)(\mathbb{R}^d) < \infty$ *and*

$$\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)\big|\mathcal{F}\Lambda(t)(\xi)\big|^2 < \infty. \tag{6.14}$$

We already met hypothesis (6.14) in Theorem 6.1. It is worthwhile to study its meaning in some important examples, like the stochastic heat and wave equations.

Lemma 6.1

1) Let $L_1 = \partial_t - \Delta_d$, where Δ_d denotes the ***Laplacian operator*** *in dimension $d \geq 1$. Then, for any $t \geq 0$, $\xi \in \mathbb{R}^d$,*

$$C_1 \frac{t}{1+|\xi|^2} \leq \int_0^t ds\big|\mathcal{F}\Lambda(s)(\xi)\big|^2 \leq C_2 \frac{t+1}{1+|\xi|^2}, \tag{6.15}$$

for some positive constants C_i, $i = 1, 2$.

Consequently (6.14) holds for $\Lambda = L_1$ if and only if

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty. \tag{6.16}$$

2) Let $L_2 = \partial^2_{tt} - \Delta_d$, $d \geq 1$. Then, for any $t \geq 0$, $\xi \in \mathbb{R}^d$, it holds that

$$c_1(t \wedge t^3) \frac{1}{1+|\xi|^2} \leq \int_0^t ds\big|\mathcal{F}\Lambda(s)(\xi)\big|^2 \leq c_2(t + t^3)\frac{1}{1+|\xi|^2}, \tag{6.17}$$

for some positive constants c_i, $i = 1, 2$.

Thus, for $\Lambda = L_2$, (6.14) is equivalent to (6.16).

Proof

In case 1), $\Lambda(t)$ is a function given by

$$\Lambda(t,x) = \big(2\pi t\big)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{2t}\right).$$

Its Fourier transform is

$$\mathcal{F}\Lambda(t)(\xi) = \exp\bigl(-2\pi^2 t|\xi|^2\bigr).$$

Hence,

$$\int_0^t ds \bigl|\mathcal{F}\Lambda(s)(\xi)\bigr|^2 = \frac{1-\exp\bigl(-4\pi^2 t|\xi|^2\bigr)}{4\pi^2|\xi|^2}.$$

On the set $\bigl(|\xi|>1\bigr)$, we have

$$\frac{1-\exp\bigl(-4\pi^2 t|\xi|^2\bigr)}{4\pi^2|\xi|^2} \le \frac{1}{2\pi^2|\xi|^2} \le \frac{C}{1+|\xi|^2}.$$

On the other hand, on $\bigl(|\xi|\le 1\bigr)$, we use the property $1-e^{-x}\le x$, $x\ge 0$, and we obtain

$$\frac{1-\exp\bigl(-4\pi^2 t|\xi|^2\bigr)}{4\pi^2|\xi|^2} \le \frac{Ct}{1+|\xi|^2}.$$

This yields the upper bound in (6.15).

Moreover, the inequality $1-e^{-x}\ge x/(1+x)$, valid for any $x\ge 0$, implies

$$\int_0^t ds \bigl|\mathcal{F}\Lambda(t)(\xi)\bigr|^2 \ge C\frac{t}{1+4\pi^2 t|\xi|^2}.$$

Assume that $4\pi^2 t|\xi|^2\ge 1$. Then $1+4\pi^2 t|\xi|^2 \le 8\pi^2 t|\xi|^2$; if $4\pi^2 t|\xi|^2\le 1$ then $1+4\pi^2 t|\xi|^2 < 2$ and therefore, $1/\bigl(1+4\pi^2 t|\xi|^2\bigr)\ge 1/\bigl(2(1+|\xi|^2)\bigr)$. Hence, we obtain the lower bound in (6.15) and now the equivalence between (6.14) and (6.16) is obvious.

Let us now consider, as in case 2), the wave operator. It is well known (see for instance ref. [64]) that

$$\mathcal{F}\Lambda(t)(\xi) = \frac{\sin\bigl(2\pi t|\xi|\bigr)}{2\pi|\xi|}.$$

Therefore

$$\begin{aligned}\bigl|\mathcal{F}\Lambda(t)(\xi)\bigr|^2 &\le \frac{1}{2\pi^2\bigl(1+|\xi|^2\bigr)}\mathbf{1}_{(|\xi|\ge 1)} + t^2\,\mathbf{1}_{(|\xi|\le 1)}\\ &\le C\frac{1+t^2}{1+|\xi|^2}.\end{aligned}$$

This yields the upper bound in (6.17).

Assume that $2\pi t|\xi| \geq 1$. Then $\sin\bigl(4\pi t|\xi|\bigr)/\bigl(2t|\xi|\bigr) \leq \pi$ and consequently,

$$\begin{aligned}\int_0^t ds \frac{\sin^2\bigl(2\pi s|\xi|\bigr)}{\bigl(2\pi|\xi|\bigr)^2} &\geq C\frac{t}{1+|\xi|^2}\int_0^{2\pi t}\sin^2\bigl(u|\xi|\bigr)\,du\\ &= C\frac{t}{1+|\xi|^2}(2\pi - \frac{\sin(4\pi t|\xi|)}{2t|\xi|})\\ &\geq C\frac{t}{1+|\xi|^2}.\end{aligned}$$

Next we assume that $2\pi t|\xi| \leq 1$ and we notice that for $r \in [0,1]$, $\sin^2 r/r^2 \geq \sin^2 1$. Thus,

$$\begin{aligned}\int_0^t ds \frac{\sin^2\bigl(2\pi s|\xi|\bigr)}{\bigl(2\pi|\xi|\bigr)^2} &\geq C\sin^2 1\int_0^{2\pi t} u^2\,du\\ &\geq C\frac{t^3}{1+|\xi|^2}.\end{aligned}$$

This finishes the proof of the lower bound in (6.17) and that of the Lemma. □

Let us now give a notion of solution to the formal equation (6.13).

Definition 6.1 A solution to the stochastic initial-value problem (6.13) with vanishing initial conditions is a *predictable* stochastic process $u = \bigl(u(t,x), (t,x) \in [0,T]\times\mathbb{R}^d\bigr)$ such that

$$\begin{aligned}\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Bigl(\bigl|u(t,x)\bigr|^2\Bigr) &< \infty,\\ E\bigl(u(t,x)u(t,y)\bigr) &= E\bigl(u(t,0)u(t,x-y)\bigr)\end{aligned} \tag{6.18}$$

and

$$\begin{aligned}u(t,x) = &\int_0^t\int_{\mathbb{R}^d}\Lambda(t-s,x-y)\sigma\bigl(u(s,y)\bigr)M(ds,dy)\\ &+\int_0^t\int_{\mathbb{R}^d} b(u(t-s,x-y))\Lambda(s,dy).\end{aligned} \tag{6.19}$$

Remark 6.3 The stochastic integral in (6.19) is of the type defined in Theorem 6.1. More precisely, here the Hilbert space $\mathcal{K}$ is $\mathbb{R}$ and $K(s,z) := \sigma\bigl(u(s,z)\bigr)$. Notice that, since σ is Lipschitz, the requirements on the process u ensure the validity of assumption B.

This setting for stochastic partial differential equations is not general enough to deal with the problem we have in mind: the study of the *probability law* of the solution of (6.13) via *Malliavin calculus*. Indeed, we need to formulate Malliavin derivatives of any order of the solution and show that they satisfy stochastic differential equations obtained by differentiation of (6.19). Hence a Hilbert-valued framework is needed.

Indeed, assume that the coefficients are differentiable; owing to (3.21), a formal differentiation of Equation (6.19) gives

$$\begin{aligned} Du(t,x) = Z(t,x) &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-y)\sigma'\big(u(s,y)\big)Du(s,y)M(ds,dy) \\ &+ \int_0^t \int_{\mathbb{R}^d} b'\big(u(t-s,x-y)\big)Du(t-s,x-y)\Lambda(s,dy), \end{aligned}$$

where $Z(t,x)$ is a $\mathcal{H}_T$-valued stochastic process that will be made explicit later.

Let $\mathcal{K}_1$, $\mathcal{K}$ be two separable Hilbert spaces. If there is no reason for misunderstanding we will use the same notation, $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, for the norms and inner products in these two spaces, respectively.

Consider two mappings

$$\sigma, b : \mathcal{K}_1 \times \mathcal{K} \longrightarrow \mathcal{K}$$

satisfying the next two conditions for some positive constant C:

(c1) $\sup_{x\in\mathcal{K}_1}\Big(\big\|\sigma(x,y)-\sigma(x,y')\big\| + \big\|b(x,y)-b(x,y')\big\|\Big) \le C\big\|y-y'\big\|$,

(c2) there exists $q \in [1,\infty)$ such that

$$\big\|\sigma(x,0)\big\| + \big\|b(x,0)\big\| \le C\big(1+\|x\|^q\big), \qquad x \in \mathcal{K}_1,\ y,y' \in \mathcal{K}.$$

Notice that (c1) and (c2) clearly imply

(c3) $\big\|\sigma(x,y)\big\| + \big\|b(x,y)\big\| \le C\big(1+\|x\|^q+\|y\|\big)$.

Let $V = \big(V(t,x),(t,x) \in [0,T]\times\mathbb{R}^d\big)$ be a predictable $\mathcal{K}_1$-valued process such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(\big\|V(t,x)\big\|^p\Big) < \infty, \tag{6.20}$$

for any $p \in [1,\infty)$, and

$$E\Big(\big\langle V(t,x),V(t,y)\big\rangle\Big) = E\Big(\big\langle V(t,0),V(t,x-y)\big\rangle\Big). \tag{6.21}$$

Consider also a predictable $\mathcal{K}$-valued process $U_0 = \big(U_0(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\big)$ satisfying the analogue of property (6.20).

Let $M_t^z(B) = M_t(z+B)$, with $z \in \mathbb{R}^d$, $B \in \mathcal{B}(\mathbb{R}^d)$ and bounded. Define $U_0^z(t,x) = U_0(t,x+z)$, $V^z(t,x) = V(t,x+z)$. We assume that the *joint distribution* of the processes $\big(U_0^z(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\big)$, $\big(V^z(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\big)$ and $\big(M_t^z(B), t \in [0,T], B \in \mathcal{B}(\mathbb{R}^d)\big)$ does not depend on z.

Set

$$\begin{aligned} U(t,x) = U_0(t,x) &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-y)\sigma\big(V(s,y), U(s,y)\big) M(ds,dy) \\ &+ \int_0^t ds \int_{\mathbb{R}^d} b\big(V(t-s,x-y), U(t-s,x-y)\big)\Lambda(s,dy). \end{aligned} \tag{6.22}$$

The next definition is the analogue of Definition 6.1 in the context of Equation (6.22).

Definition 6.2 A solution to Equation (6.22) is a $\mathcal{K}$-valued predictable stochastic process $\big(U(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\big)$ such that

(a) $\displaystyle\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(\big\|U(t,x)\big\|^2\Big) < \infty.$

(b) $E\Big(\big\langle U(t,x), U(t,y)\big\rangle\Big) = E\Big(\big\langle U(t,0), U(t,x-y)\big\rangle\Big)$

and it satisfies the relation (6.22).

Remark 6.4 Since we are assuming Hypothesis D, the stochastic integral in (6.22) is of the type given in Theorem 6.1. Indeed, condition (c3) on the coefficients yields

$$\begin{aligned} &\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(\Big\|\sigma\big(V(t,x), U(t,x)\big)\Big\|^2\Big) \\ &\qquad \le \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} CE\Big(1 + \big\|U(t,x)\big\|^2 + \big\|V(t,x)\big\|^{2q}\Big) \le C. \end{aligned}$$

The constant C is finite (see condition (a) in the previous definition and (6.20)). Hence Hypothesis B is satisfied.

The validity of Hypothesis C can be checked following similar arguments as those of the proof of Lemma 18 in [14].

Our next purpose is to prove a result on existence and uniqueness of solution for the Equation (6.22). In particular we shall obtain the version proved in [14] for the particular case of Equation (6.19)

Theorem 6.2

We assume that the coefficients σ and b satisfy the conditions (c1) and (c2) above and moreover, that Hypothesis D is satisfied. Then, Equation (6.22) has a unique solution in the sense given in Definition 6.2.

In addition the solution satisfies

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(\big\|U(t,x)\big\|^p\Big) < \infty, \tag{6.23}$$

for any $p \in [1,\infty)$.

PROOF

Define a *Picard iteration scheme*, as follows.

$U^0(t,x) = U_0(t,x)$,

and for $n \geq 1$

$$\begin{aligned} U^n(t,x) &= U_0(t,x) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-y)\sigma\big(V(s,y),U^{n-1}(s,y)\big)M(ds,dy) \\ &\quad + \int_0^t ds \int_{\mathbb{R}^d} b\big(V(t-s,x-y),U^{n-1}(t-s,x-y)\big)\Lambda(s,dy), \end{aligned} \tag{6.24}$$

Fix $p \in [1,\infty)$. We prove the following facts:

(i) $U^n = \big(U^n(t,x),(t,x) \in [0,T] \times \mathbb{R}^d\big)$, $n \geq 1$, are well defined predictable process and have spatially stationary covariance.

(ii) $\sup_{n\geq 0} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(\big\|U^n(t,x)\big\|^p\Big) < \infty$.

(iii) Set $M_n(t) = \sup_{x\in\mathbb{R}^d} E\Big(\big\|U^{n+1}(t,x) - U^n(t,x)\big\|^p\Big)$, $n \geq 0$. Then

$$M_n(t) \leq C \int_0^t ds M_{n-1}(s)\big(J(t-s)+1\big), \tag{6.25}$$

where

$$J(t) = \int_{\mathbb{R}^d} \mu(d\xi)\big|\mathcal{F}\Lambda(t)(\xi)\big|^2. \tag{6.26}$$

PROOF OF (i)

We prove by induction on n that U^n is predictable and

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(\big\|U^n(t,x)\big\|^2\Big) < \infty. \tag{6.27}$$

This suffices to give a rigourous meaning to the integrals appearing in (6.24). Indeed, by assumption this is true for $n = 0$. Assume that the property is true for any $k = 0, 1, \dots n-1$, $n \geq 2$. Consider the stochastic process given by

$$K(t,x) = \sigma\big(V(x,t), U^{n-1}(t,x)\big). \tag{6.28}$$

The induction assumption and the arguments of Remark 6.4 ensure that the assumptions of Theorem 6.1 are satisfied. In particular, (6.8) for $p = 2$ yields

$$\begin{aligned} &\sup_{\substack{(t,x)\in\\ [0,T]\times\mathbb{R}^d}} E\left(\left\|\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-y)\sigma\big(V(s,y),U^{n-1}(s,y)\big)M(ds,dy)\right\|^2\right)\\ &\quad\leq \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} CE\Big(1+\big\|U^{n-1}(t,x)\big\|^2+\big\|V(t,x)\big\|^{2q}\Big)\\ &\qquad\quad \times \int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)\big|\mathcal{F}\Lambda(t)(\xi)\big|^2. \end{aligned} \tag{6.29}$$

This last expression is finite, by assumption.

Similarly,

$$\begin{aligned} &\sup_{\substack{(t,x)\in\\ [0,T]\times\mathbb{R}^d}} E\left(\left\|\int_0^t ds \int_{\mathbb{R}^d} b\big(V(t-s,x-y),U^{n-1}(t-s,x-y)\big)\Lambda(s,dy)\right\|^2\right)\\ &\quad\leq \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} CE\Big(1+\big\|U^{n-1}(t,x)\big\|^2+\big\|V(t,x)\big\|^{2q}\Big)\int_0^T ds\,\Lambda(s,\mathbb{R}^d), \end{aligned} \tag{6.30}$$

which is also finite. Hence we deduce (6.27).

The property on the covariance is also proved inductively with the arguments of Lemma 18 in [14].

Proof of (ii)

Fix $p \in [1, \infty)$. We first prove that for any $n \geq 1$,

$$\sup_{x\in\mathbb{R}^d} E\Big(\big\|U^n(t,x)\big\|^p\Big) \\ \leq C_1 + C_2 \int_0^t ds \left(\sup_{x\in\mathbb{R}^d} E\Big(\big\|U^{n-1}(s,x)\big\|^p\Big)\big(J(t-s)+1\big)\right) \tag{6.31}$$

$t \in [0,T]$, $n \geq 1$. The arguments are not very far from those used in the proof of (i).

Indeed, we have

$$E\Big(\big\|U^n(t,x)\big\|^p\Big) \leq C\big(C_0(t,x) + A_n(t,x) + B_n(t,x)\big), \tag{6.32}$$

with

$$C_0(t,x) = E\Big(\big\|U_0(t,x)\big\|^p\Big),$$
$$A_n(t,x) = E\left(\left\|\int_0^t\int_{\mathbb{R}^d} \Lambda(t-s,x-y)\sigma\big(V(s,y),U^{n-1}(s,y)\big)M(ds,dy)\right\|^p\right),$$
$$B_n(t,x) = E\left(\left\|\int_0^t\int_{\mathbb{R}^d} b\big(V(t-s,x-y),U^{n-1}(t-s,x-y)\big)\Lambda(s,dy)\right\|^p\right).$$

By assumption

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} C_0(t,x) < \infty. \tag{6.33}$$

Consider the stochastic process $K(t,x)$ defined in (6.28), which satisfies the assumptions of Theorem 6.1. In particular (6.8) yields

$$\sup_{x\in\mathbb{R}^d} A_n(t,x) \leq C\int_0^t ds\left(1 + \sup_{y\in\mathbb{R}^d} E\Big(\big\|U^{n-1}(s,y)\big\|^p\Big)\right) \\ \times \int_{\mathbb{R}^d} \mu(d\xi)\big|\mathcal{F}\Lambda(t-s)(\xi)\big|^2. \tag{6.34}$$

Moreover, *Jensen's inequality* implies

$$
\begin{aligned}
\sup_{x\in\mathbb{R}^d} B_n(t,x)&\\
&\le C\int_0^t ds\left(1+\sup_{y\in\mathbb{R}^d} E\Big(\big\|U^{n-1}(s,y)\big\|^p\Big)\right)\int_{\mathbb{R}^d}\Lambda(t-s,dy)\\
&= C\int_0^t ds\left(1+\sup_{y\in\mathbb{R}^d} E\Big(\big\|U^{n-1}(s,y)\big\|^p\Big)\right)\Lambda(t-s,\mathbb{R}^d)\\
&\le C\int_0^t ds\left(1+\sup_{y\in\mathbb{R}^d} E\Big(\big\|U^{n-1}(s,y)\big\|^p\Big)\right).
\end{aligned}
\tag{6.35}
$$

Plugging the estimates (6.33) to (6.35) into (6.32) yields (6.31).

Finally, the conclusion of part (ii) follows applying the version of Gronwall's Lemma given in Lemma 6.2 below to the following situation: $f_n(t)=\sup_{x\in\mathbb{R}^d} E\big(\|U^n(t,x)\|^p\big)$, $k_1=C_1$, $k_2=0$, $g(s)=C_2(J(s)+1)$, with C_1,C_2 given in (6.31).

Proof of (iii)

Consider the decomposition

$$
E\Big(\big\|U^{n+1}(t,x)-U^n(t,x)\big\|^p\Big)\le C\big(a_n(t,x)+b_n(t,x)\big),
$$

with

$$
\begin{aligned}
a_n(t,x) &= E\bigg(\bigg\|\int_0^t\int_{\mathbb{R}^d}\Lambda(t-s,x-y)\\
&\qquad\times\Big(\sigma\big(V(s,y),U^n(s,y)\big)-\sigma\big(V(s,y),U^{n-1}(s,y)\big)\Big)M(ds,dy)\bigg\|^p\bigg),\\
b_n(t,x) &= E\bigg(\bigg\|\int_0^t ds\int_{\mathbb{R}^d}\Lambda(s,dy)\Big(b\big(V(t-s,x-y),U^n(t-s,x-y)\big)\\
&\qquad - b\big(V(t-s,x-y),U^{n-1}(t-s,x-y)\big)\Big)\bigg\|^p\bigg).
\end{aligned}
$$

Then (6.25) follows by similar arguments as those which lead to (6.31), using the *Lipschitz* condition (c1).

We finish the proof applying Lemma 6.2 in the particular case: $f_n(t)=M_n(t)$, $k_1=k_2=0$, $g(s)=C(J(s)+1)$, with C given in (6.25). Notice that the results proved in part (ii) show that $M:=\sup_{0\le s\le T} f_0(s)$ is finite.

Then we conclude that $(U^n(t,x),(t,x) \in [0,T] \times \mathbb{R}^d)$ converges uniformly in $L^p(\Omega)$ to a limit $U = \big(U(t,x),(t,x) \in [0,T] \times \mathbb{R}^d\big)$. It is not difficult to check that U satisfies the conditions of Definition 6.2 and therefore the theorem is completely proved. □

Examples 6.1

Let $\mathcal{K} = \mathcal{A} = \mathbb{R}$, σ and b depending only on the second variable $y \in \mathbb{R}$. Then condition (c1) states the Lipschitz continuity, (c2) is trivial and (c3) follows from (c1). Equation (6.22) is of the same kind as (6.19), except for the non trivial initial condition. Therefore Theorem 6.2 yields the existence of a unique solution in the sense of Definition 6.1. Moreover, the process u satisfies

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(\big|u(t,x)\big|^p\Big) < \infty. \tag{6.36}$$

This is a variant of Theorem 13 in reference [14].

We finish this section quoting a technical result — a version of *Gronwall's Lemma* proved in reference [14] — that has been applied in the proof of the previous theorem.

Lemma 6.2

(Lemma 15 of ref. [14]) Let $g : [0,T] \to \mathbb{R}_+$ be a non-negative function such that $\int_0^T g(s)ds < \infty$. Then, there is a sequence $(a_n,\, n \in \mathbb{N})$ of non-negative real numbers such that for all $p \geq 1$, $\sum_{n=1}^\infty a_n^{1/p} < \infty$, and having the following property:

Let $(f_n, n \in \mathbb{N})$ be a sequence of non-negative functions on $[0,T]$, k_1, k_2 be non-negative numbers such that for $0 \leq t \leq T$,

$$f_n(t) \leq k_1 + \int_0^t \big(k_2 + f_{n-1}(s)\big) g(t-s)\, ds.$$

If $\sup_{0\leq s\leq T} f_0(s) = M$, then for $n \geq 1$,

$$f_n(t) \leq k_1 + (k_1 + k_2) \sum_{i=1}^{n-1} a_i + (k_2 + M)a_n.$$

In particular, $\sup_{n\geq 0} \sup_{0\leq t\leq T} f_n(t) < \infty$; if $k_1 = k_2 = 0$, then $\sum_{n\geq 0} f_n(t)^{1/p}$ converges uniformly on $[0,T]$.

Comments

This chapter requires as prerequisite knowledge of the theory of stochastic integration with respect to martingale measures, as has been developed in reference [67].

The results of Section 6.1 are from reference [54] and are deeply inspired on reference [14]. Theorem 6.2 is a generalized version of a slight variant of Theorem 13 in reference [14]. The analysis of Hypothesis D owns to work published in references [25], [32] and [35].

A different approach to SPDE's driven by coloured noise and the study of the required stochastic integrals is given in reference [53] (see also the references therein), following mainly the theoretical basis from reference [16].

6.3 Exercises

6.3.1

Let S be a distribution-valued function defined on $[0,T]$ satisfying the conditons of Theorem 6.1. Prove the following statements.

1) Assume that there exist constants $C>0$ and $\gamma_1 \in (0.\infty)$ such that

$$\int_{t_1}^{t_2} ds \int_{\mathbb{R}^d} \mu(d\xi)\big|\mathcal{F}S(s)(\xi)\big|^2 \le C\big|t_2-t_1\big|^{2\gamma_1}, \tag{6.37}$$

$0 \le t_1 \le t_2 \le T$. Then for any $p \in [2,\infty)$, $T>0$, $h \ge 0$,

$$\sup_{0\le t\le T} E\Big(\big|(S\cdot M)_{t+h} - (S\cdot M)_t\big|^p\Big) \le Ch^{\gamma_1 p}. \tag{6.38}$$

Hence the stochastic process $\big((S\cdot M)_t, t\in[0,T]\big)$ has a.s. *α-Hölder continuous paths* for any $\alpha \in (0,\gamma_1)$.

2) Assume (6.37) and in addition that there exist a constant $C>0$ and $\gamma_2 \in (0,\infty)$ such that

$$\int_0^T \int_{\mathbb{R}^d} \mu(d\xi)\big|\mathcal{F}S(s+h)(\xi) - \mathcal{F}S(s)(\xi)\big|^2 \le Ch^{2\gamma_2}. \tag{6.39}$$

Then, for any $p\in[2,\infty)$, $h\ge 0$,

$$\sup_{0\le t\le T} E\Big(\Big|\big(S(t+h-\cdot)\cdot M\big)_{t+h} - \big(S(t-\cdot)\cdot M\big)_t\Big|^p\Big) \le Ch^{\gamma p}, \tag{6.40}$$

with $\gamma = \min(\gamma_1,\gamma_2)$. Hence the stochastic process $\big((S(t-\cdot)\cdot M)_t, t\ge 0\big)$ has a.s. β-Hölder continuous paths for any $\beta\in(0,\gamma)$.

Hint: Here the stochastic integrals define Gaussian processes. Therefore L^p estimates follow from L^2 estimates. The former are obtained using the isometry property of the stochastic integral (see (6.7) with $\mathcal{K} = \mathbb{R}$ and $K = 1$).

6.3.2

Suppose there exists $\eta \in (0, 1)$ such that

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{\left(1 + |\xi|^2\right)^\eta} < \infty. \tag{6.41}$$

Prove that the above conditions (6.37), (6.39) are satisfied by

1) the fundamental solution of the wave equation, with $\gamma_1 \in (0, \frac{1}{2})$ and $\gamma_2 \in (0, 1 - \eta]$,
2) the fundamental solution of the heat equation, with $\gamma_j \in (0, \frac{1-\eta}{2}]$, $j = 1, 2$.

Hint: Split the integral with respect to the variable in $\mathbb{R}^d$ into two parts: a neighbourhood of the origin and the complementary set.

6.3.3

Let S be as in Exercise 6.3.1. Assume that for any compact set $K \subset \mathbb{R}^d$ there exists $\gamma \in (0, \infty)$ and $C > 0$ such that

$$\int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) \left|\mathcal{F}S(s, x + z - \cdot)(\xi) - \mathcal{F}S(s, x - \cdot)(\xi)\right|^2 \le C|z|^{2\gamma},$$

$x \in \mathbb{R}^d$, $z \in K$. Prove that for any $x \in \mathbb{R}^d$, $z \in K$, $p \in [2, \infty)$, there exists a positive constant C such that

$$\sup_{0 \le t \le T} E\Big(\Big|\big(S(t - \cdot, x + z - \cdot) \cdot M\big)_t - \big(S(t - \cdot, x - \cdot) \cdot M\big)_t\Big|^p\Big) \le C|z|^{p\gamma}. \tag{6.42}$$

Thus, the process $\big((S(t - \cdot, x - \cdot) \cdot M)_t\big)$ has a.s. β-Hölder continuous paths, with $\beta \in (0, \gamma)$.

6.3.4

Assume that condition (6.41) holds. Prove that the estimate (6.42) holds true for the fundamental solutions of the heat and the wave equation with $\gamma \in (0, 1 - \eta)$.

6.3.5

Consider the *stochastic heat equation* in dimension $d \ge 1$ with null initial condition. That is, Equation (6.19) with $\Lambda(t, x) = (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right)$. The aim of this exercise is to prove that if condition (6.41) is satisfied, then the paths of the solution are β_1-Hölder continuous in t and β_2-Hölder continuous in x with $\beta_1 \in (0, \frac{1-\eta}{2})$, $\beta_2 \in (0, 1-\eta)$.

The proof uses the *factorization method* (see ref. [16]) and can be carried out following the next steps.

1) Fix $\alpha \in (0,1)$ and set

$$Y_\alpha(r,z) = \int_0^r \int_{\mathbb{R}^d} \Lambda(r-s, z-y)\sigma\big(u(s,y)\big)(r-s)^{-\alpha} M(ds,dy).$$

Using the semigroup property of Λ and the *stochastic Fubini's theorem* from reference [67], prove that

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^d} &\Lambda(t-s, x-y)\sigma\big(u(s,y)\big) M(ds,dy) \\ &= \frac{\sin(\pi\alpha)}{\pi} \int_0^t dr \int_{\mathbb{R}^d} dz \Lambda(t-r, x-z)(t-r)^{\alpha-1} Y_\alpha(r,z).\end{aligned} \tag{6.43}$$

2) Check that for any $p \in [1,\infty)$, $\alpha \in (0, \frac{1-\eta}{2})$,

$$\sup_{0\le r\le T} \sup_{z\in\mathbb{R}^d} E\Big(\big|Y_\alpha(r,z)\big|^p\Big) < \infty.$$

3) Using *Kolmogorov's continuity criterium* and the previous result, prove the estimates

$$\sup_{0\le t\le T} \sup_{x\in\mathbb{R}^d} E\Big(\big|u(t+h,x) - u(t,x)\big|^p\Big) \le C h^{\gamma_1 p},$$

$$\sup_{0\le t\le T} \sup_{x\in\mathbb{R}^d} E\Big(\big|u(t,x+z) - u(t,x)\big|^p\Big) \le C|z|^{\gamma_2 p},$$

$t, h \in [0,T]$, $t+h \in [0,T]$, $z \in \mathbb{R}^d$, $\gamma_1 \in (0, \frac{1-\eta}{2})$, $\gamma_2 \in (0, 1-\eta)$.

Hints: To prove part 2 apply the L^p estimates of the stochastic integral given in Theorem 6.1. Then the problem reduces to the check that

$$\nu_{r,z} = \int_0^r ds \int_{\mathbb{R}^d} \mu(d\xi) \big|\mathcal{F}[\Lambda(r-s, z-\cdot)(r-s)^{-\alpha}](\xi)\big|^2$$

is finite. This can be proved splitting the integral on $\mathbb{R}^d$ into two parts — in a neighbourhood and outside zero. The proof of the estimates in part 3 is carried out using the alternative expression of the stochastic integral given in (6.43), Hölder's inequality and the result of part 2.

Remark. The exercises of this section are excerpts of references [59] and [60].

Chapter 7

Malliavin Regularity of Solutions of SPDE's

In this chapter we show that if σ and b are smooth, then the solution of equation (6.13) at any fixed point $(t,x) \in [0,T] \times \mathbb{R}^d$ belongs to the space $\mathbb{D}^\infty$ and we deduce the equation satisfied by the Malliavin derivative of any order.

Following the discusion at the begining of Chapter 3, the underlying Gaussian family needed in the Malliavin calculus machinery shall be $\big(W(h), h \in \mathcal{H}_T\big)$, where for any $h \in \mathcal{H}_T$, $W(h) = \sum_{n\geq 1}\langle h, e_n\rangle_{\mathcal{H}_T} g_n$, $(e_n, n \geq 1)$ is a complete orthonormal system of $\mathcal{H}_T$ and $(g_n, n \geq 1)$ a sequence of standard independent Gaussian random variables. Actually $W(h)$ can be considered as an stochastic integral in Dalang's sense, as in reference [14], of a deterministic integrand $h \in \mathcal{H}_T$ with respect to the martingale measure M introduced in Chapter 7. Indeed for any $h \in \mathcal{H}_T$ there exists a sequence $(h_n, n \geq 1) \subset L^2\big([0,T]; \mathcal{E}\big)$ converging to h in the topology of $\mathcal{H}_T$. Set $W(h_n) = \int_0^T \int_{\mathbb{R}^d} h_n(s,x) M(ds,dx)$. The stochastic integral is well defined as a Walsh integral of a deterministic function with respect to the martingale measure M.

Notice that $E\big(W(h_n)\big) = 0$ and

$$E\big(W(h_n)W(h_m)\big) = \int_0^T \int_{\mathbb{R}^d} \mu(d\xi)\mathcal{F}h_n(s)(\xi)\overline{\mathcal{F}h_m(s)(\xi)}.$$

Set

$$\tilde{W}(h) = \lim_{n\to\infty} W(h_n),$$

in $L^2(\Omega, P)$. Then $\big(\tilde{W}(h), h \in \mathcal{H}_T\big)$ is a Gaussian family of random variables with the same characteristics (mean and covariance operators) as $\big(W(h), h \in \mathcal{H}_T\big)$.

The proof of differentiability uses the following tool which follows from the fact that D^N is a closed operator defined on $L^p(\Omega)$ with values in $L^p(\Omega; \mathcal{H}_T^{\otimes N})$.

Lemma 7.1

Let $(F_n, n \geq 1)$ be a sequence of random variables belonging to $\mathbb{D}^{N,p}$. Assume that:

(a) there exists a random variable F such that F_n converges to F in $L^p(\Omega)$, as n tends to ∞,

(b) the sequence $(D^N F_n, n \geq 1)$ converges in $L^p(\Omega; \mathcal{H}_T^{\otimes N})$.

Then F belongs to $\mathbb{D}^{N,p}$ and $D^N F = L^p(\Omega; \mathcal{H}_T^{\otimes N}) - \lim_{n\to\infty} D^N F_n$.

Before stating the main result, we introduce some notation. For $r_i \in [0,T]$, $\varphi_i \in \mathcal{H}$, $i = 1, \dots, N$ and a random variable X we set

$$D^N_{((r_1,\varphi_1),\dots,(r_N,\varphi_N))} X = \left\langle D^N_{(r_1,\dots,r_N)} X, \varphi_1 \otimes \cdots \otimes \varphi_N \right\rangle_{\mathcal{H}^{\otimes N}}.$$

Thus, we have that

$$\left\| D^N X \right\|^2_{\mathcal{H}_T^{\otimes N}} = \int_{[0,T]^N} dr_1 \cdots dr_N \sum_{j_1,\dots,j_N \geq 0} \left| D^N_{((r_1,e_{j_1}),\dots,(r_N,e_{j_N}))} X \right|^2, \tag{7.1}$$

where $\{e_j\}_{j\geq 0}$ is a complete orthonormal system of $\mathcal{H}$.

Let $N \in \mathbb{N}$, fix a set $A_N = \{\alpha_i = (r_i, \varphi_i) \in \mathbb{R}_+ \times \mathcal{H},\ i = 1, \dots, N\}$ and set $\bigvee_i r_i = \max(r_1, \dots, r_N)$, $\alpha = (\alpha_1, \dots, \alpha_N)$, $\hat{\alpha}_i = (\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_N)$. Denote by $\mathcal{P}_m$ the set of partitions of A_N consisting of m disjoint subsets $p_1, \dots, p_m$, $m = 1, \dots, N$, and by $|p_i|$ the cardinal of p_i. Let X be a random variable belonging to $\mathbb{D}^{N,2}$, $N \geq 1$, and g be a real $\mathcal{C}^N$ function with bounded derivatives up to order N. Leibniz's rule for Malliavin's derivatives (see (3.20)) yields

$$D^N_\alpha \big(g(X)\big) = \sum_{m=1}^N \sum_{\mathcal{P}_m} c_m g^{(m)}(X) \prod_{i=1}^m D^{|p_i|}_{p_i} X, \tag{7.2}$$

with positive coefficients c_m, $m = 1, \dots, N$, $c_1 = 1$.

Let

$$\triangle^N_\alpha(g, X) := D^N_\alpha \big(g(X)\big) - g'(X) D^N_\alpha X.$$

Notice that $\triangle^N_\alpha(g, X) = 0$ if $N = 1$, and for any $N > 1$ it only depends on the Malliavin derivatives up to the order $N - 1$.

Here is the result on differentiability of the process $\big(u(t,x),(t,x)\in[0,T]\times\mathbb{R}^d\big)$ solution to (6.19).

Theorem 7.1

Assume Hypothesis D and that the coefficients σ and b are $\mathcal{C}^\infty$ functions with bounded derivatives of any order greater or equal than one. Then, for every $(t,x)\in[0,T]\times\mathbb{R}^d$, the random variable $u(t,x)$ belongs to the space $\mathbb{D}^\infty$.

Moreover, for any $p\geq 1$ and $N\geq 1$, there exists a $L^p\big(\Omega;\mathcal{H}_T^{\otimes N}\big)$-valued random process $\big\{Z^N(t,x),(t,x)\in[0,T]\times\mathbb{R}^d\big\}$ such that

$$\begin{aligned} D^N u(t,x) = Z^N(t,x) + \int_0^t\int_{\mathbb{R}^d} \Lambda(t-s,x-z)\Big[\triangle^N\big(\sigma,u(s,z)\big) \\ + D^N u(s,z)\sigma'\big(u(s,z)\big)\Big] M(ds,dz) \\ + \int_0^t ds\int_{\mathbb{R}^d}\Lambda(s,dz)\Big[\triangle^N\big(b,u(t-s,x-z)\big) \\ + D^N u(t-s,x-z)b'\big(u(t-s,x-z)\big)\Big], \end{aligned} \tag{7.3}$$

and

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}^d} E\Big(\big\|D^N u(s,y)\big\|^p_{\mathcal{H}_T^{\otimes N}}\Big) < +\infty.$$

The proof of this Theorem consists of two parts. In the first one we shall assume that the measure on $\mathcal{B}(\mathbb{R}^d)$, $\Lambda(t)$, is absolutely continuous with respect to the Lebesgue measure, that is,

$$\Lambda(t,dx) = \Lambda(t,x)\,dx. \tag{7.4}$$

In the second one, we shall consider a mollifying procedure, use the results obtained in the first part and prove the result in the more general case of a non-negative measure. The next Proposition is devoted to the first part.

Proposition 7.1

Assume Hypothesis D and, in addition, that the measure $\Lambda(t)$ is absolutely continuous with respect to the Lebesgue measure. Suppose that the coefficients σ and b are $\mathcal{C}^\infty$ functions with bounded derivatives of any order greater or equal than one. Then, for every $(t,x)\in[0,T]\times\mathbb{R}^d$, the random variable $u(t,x)$ belongs to the space $\mathbb{D}^\infty$.

Moreover, for any $N \geq 1$, the Malliavin derivative $D^N u(t,x)$ satisfies the equation

$$\begin{aligned} D^N u(t,x) = Z^N(t,x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z) \big[\triangle^N\big(\sigma,u(s,z)\big) \\ +D^N u(s,z)\sigma'\big(u(s,z)\big)\big] M(ds,dz) \\ + \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz) \big[\triangle^N\big(b,u(t-s,x-z)\big) \\ +D^N u(t-s,x-z) b'\big(u(t-s,x-z)\big)\big], \end{aligned} \tag{7.5}$$

where, for $\alpha = \big((r_1,\varphi_1),\dots,(r_N,\varphi_N)\big)$ with $r_1,\dots,r_n \geq 0$ and $\varphi_1,\dots,\varphi_N \in \mathcal{H}$,

$$Z^N_\alpha(t,x) = \sum_{i=1}^N \Big\langle \Lambda(t-r_i,x-*) D^{N-1}_{\hat{\alpha}_i} \sigma\big(u(r_i,*)\big), \varphi_i \Big\rangle_{\mathcal{H}}. \tag{7.6}$$

In addition, for any $p \in [1,\infty)$,

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}^d} E\Big(\big\|D^N u(s,y)\big\|^p_{\mathcal{H}_T^{\otimes N}}\Big) < +\infty.$$

The proof of this Proposition relies on the results given in the next three lemmas based on the Picard iterations

$$\begin{aligned} u^0(t,x) &= 0, \\ u^n(t,x) &= \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-y) \sigma\big(u^{n-1}(s,y)\big) M(ds,dy) \\ &\quad + \int_0^t ds \int_{\mathbb{R}^d} b\big(u^{n-1}(t-s,x-y)\big) \Lambda(s,dy). \end{aligned} \tag{7.7}$$

Lemma 7.2

Under the hypothesis of Proposition 7.1, the sequence of random variables $\big(u^n(t,x), n \geq 0\big)$ defined recursively in (7.7) belong to $\mathbb{D}^{N,2}$.

Proof

It is done by a recursive argument on N. Let $N = 1$. We check that $u^n(t,x) \in \mathbb{D}^{1,2}$, for any $n \geq 0$.

Clearly the property is true for $n = 0$. Assume it holds up to the $(n-1)$-th iteration. By the rules of Malliavin calculus, the right hand-side of (7.7) belongs to $\mathbb{D}^{1,2}$. Hence $u^n(t,x) \in \mathbb{D}^{1,2}$, and moreover

$$\begin{aligned} &Du^n(t,x) \\ &= \Lambda(t-\cdot, x-*)\sigma\big(u^{n-1}(\cdot,*)\big) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-y)\sigma'\big(u^{n-1}(s,y)\big)Du^{n-1}(s,y)M(ds,dy) \\ &\quad + \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dy)b'\big(u^{n-1}(t-s,x-y)\big)Du^{n-1}(t-s,x-y). \end{aligned} \tag{7.8}$$

Assume that $u^n(t,x) \in \mathbb{D}^{N-1,2}$, for any $n \geq 0$. Leibniz's rule and (3.21) yield the following equality satisfied by $D^{N-1}u^n(t,x)$,

$$\begin{aligned} &D_\alpha^{N-1}u^n(t,x) \\ &\quad = \sum_{i=1}^{N-1} \Big\langle \Lambda(t-r_i, x-*)D_{\hat{\alpha}_i}^{N-2}\sigma\big(u^{n-1}(r_i,*)\big), \varphi_i \Big\rangle_{\mathcal{H}} \\ &\quad + \int_{\bigvee_i r_i}^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z)\left[\triangle_\alpha^{N-1}(\sigma, u^{n-1}(s,z)\right) \\ &\quad\quad + D_\alpha^{N-1}u^{n-1}(s,z)\sigma'\big(u^{n-1}(s,z)\big)\Big]\, M(ds,dz) \\ &\quad + \int_{\bigvee_i r_i}^t ds \int_{\mathbb{R}^d} \Lambda(s,dz)\left[\triangle_\alpha^{N-1}\big(b, u^{n-1}(t-s,x-z)\big)\right. \\ &\quad\quad \left. + D_\alpha^{N-1}u^{n-1}(t-s,x-z)b'\big(u^{n-1}(t-s,x-z)\big)\right], \end{aligned} \tag{7.9}$$

where $\alpha = \big((r_1,\varphi_1),\dots,(r_{N-1},\varphi_{N-1})\big)$, with $r_1,\dots,r_{N-1} \geq 0$ and $\varphi_1,\dots,\varphi_{N-1} \in \mathcal{H}$. We want to prove that $u^n(t,x) \in \mathbb{D}^{N,2}$, for any $n \geq 0$ as well. Clearly the property is true for $n = 0$. Assume it holds for all the iterations up to the order $n-1$. Then, as before, using the rules of Malliavin calculus we obtain that the right hand side of the preceding equality belongs to $\mathbb{D}^{N,2}$. Thus, $u^n(t,x) \in \mathbb{D}^{N,2}$ and satisfies

$$\begin{aligned} &D_\alpha^{N}u^n(t,x) \\ &\quad = \sum_{i=1}^{N} \Big\langle \Lambda(t-r_i, x-*)D_{\hat{\alpha}_i}^{N-1}\sigma\big(u^{n-1}(r_i,*)\big), \varphi_i \Big\rangle_{\mathcal{H}} \\ &\quad + \int_{\bigvee_i r_i}^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z)\left[\triangle_\alpha^{N}\big(\sigma, u^{n-1}(s,z)\right) \\ &\quad\quad + D_\alpha^{N}u^{n-1}(s,z)\sigma'\big(u^{n-1}(s,z)\big)\Big]\, M(ds,dz) + \end{aligned} \tag{7.10}$$

$$+\int_{\bigvee_i r_i}^{t} ds \int_{\mathbb{R}^d} \Lambda(s,dz) \left[\triangle_\alpha^N\big(b, u^{n-1}(t-s,x-z)\big)\right.$$
$$\left.+D_\alpha^N u^{n-1}(t-s,x-z) b'\big(u^{n-1}(t-s,x-z)\big)\right],$$

where $\alpha = \big((r_1,\varphi_1),\dots,(r_N,\varphi_N)\big)$, with $r_1,\dots,r_N \geq 0$ and $\varphi_1,\dots,\varphi_N \in \mathcal{H}$.

This ends the proof of the Lemma. □

Lemma 7.3

Assume the same hypothesis than in Proposition 7.1. Then, for any positive integer $N \geq 1$ and for all $p \in [1,\infty)$,

$$\sup_{n\geq 0} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(\|D^N u^n(t,x)\|^p_{\mathcal{H}_T^{\otimes N}}\Big) < \infty. \tag{7.11}$$

PROOF

We shall use an induction argument with respect to N with $p \geq 2$ fixed. Consider $N=1$. Denote by $B_{i,n}$, $i=1,2,3$ each one of the terms on the right hand-side of (7.8), respectively. Hölder's inequality with respect to the finite measure $\Lambda(t-s,x-y)\Lambda(t-s,x-y+z)\,ds\,\Gamma(dz)\,dy$, the Cauchy-Schwarz inequality and the properties of σ imply

$$\begin{aligned}
&E\Big(\|B_{1,n}\|^p_{\mathcal{H}_T}\Big)\\
&\quad= E\bigg(\int_0^t ds \int_{\mathbb{R}^d}\Gamma(dz)\int_{\mathbb{R}^d} dy \Lambda(t-s,x-y)\Lambda(t-s,x-y+z)\\
&\qquad\times \sigma\big(u^{n-1}(s,y)\big)\sigma\big(u^{n-1}(s,y-z)\big)\bigg)^{\frac{p}{2}}\\
&\quad\leq \bigg(\int_0^t ds \int_{\mathbb{R}^d}\Gamma(dz)\int_{\mathbb{R}^d} dy \Lambda(t-s,x-y)\Lambda(t-s,x-y+z)\bigg)^{\frac{p}{2}-1}\\
&\qquad\times \int_0^t ds \int_{\mathbb{R}^d}\Gamma(dz)\int_{\mathbb{R}^d} dy \Lambda(t-s,x-y)\Lambda(t-s,x-y+z)\\
&\qquad\times E\Big(\big|\sigma\big(u^{n-1}(s,y)\big)\sigma\big(u^{n-1}(s,y-z)\big)\big|^{\frac{p}{2}}\Big)\\
&\quad\leq C\bigg(1+\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(\big|u^{n-1}(t,x)\big|^p\Big)\bigg)\int_0^T ds \int_{\mathbb{R}^d}\mu(d\xi)\big|\mathcal{F}\Lambda(s)(\xi)\big|^2,
\end{aligned}$$

which is uniformly bounded with respect to n (see (ii) in the proof of Theorem 6.2).

Consider now the second term $B_{2,n}(t,x)$. By Theorem 6.1 and the properties of σ, we have

$$\begin{aligned}
&E\Big(\big\|B_{2,n}(t,x)\big\|^p_{\mathcal{H}_T}\Big)\\
&\quad\le C\int_0^t ds\sup_{z\in\mathbb{R}^d}E\bigg(\Big\|\sigma'\big(u^{n-1}(s,z)\big)Du^{n-1}(s,z)\Big\|^p_{\mathcal{H}_T}\bigg)J(t-s)\\
&\quad\le C\int_0^t ds\sup_{(\tau,z)\in[0,s]\times\mathbb{R}^d}E\Big(\big\|Du^{n-1}(\tau,z)\big\|^p_{\mathcal{H}_T}\Big)J(t-s),
\end{aligned}$$

with J defined as in (6.26).

Finally, for the third term $B_{3,n}(t,x)$ we use Hölder's inequality with respect to the finite measure $\Lambda(s,dz)ds$. Then, the assumptions on b and Λ yield

$$E\Big(\big\|B_{3,n}(t,x)\big\|^p_{\mathcal{H}_T}\Big)\le C\int_0^t ds\sup_{(\tau,z)\in[0,s]\times\mathbb{R}^d}E\Big(\big\|Du^{n-1}(\tau,z)\big\|^p_{\mathcal{H}_T}\Big).$$

Therefore,

$$\begin{aligned}
&\sup_{(s,z)\in[0,t]\times\mathbb{R}^d}E\Big(\big\|Du^n(s,z)\big\|^p_{\mathcal{H}_T}\Big)\\
&\quad\le C\left(1+\int_0^t ds\sup_{(\tau,z)\in[0,s]\times\mathbb{R}^d}E\Big(\big\|Du^{n-1}(\tau,z)\big\|^p_{\mathcal{H}_T}\Big)\big(J(t-s)+1\big)\right).
\end{aligned}$$

Then, by Gronwall's Lemma 6.2 we finish the proof for $N=1$.

Assume that

$$\sup_{n\ge 0}\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}E\Big(\big\|D^k u^n(t,x)\big\|^p_{\mathcal{H}_T^{\otimes k}}\Big)<+\infty,$$

for any $k=1,\dots,N-1$. Let $\alpha=\big((r_1,e_{j_1}),\dots,(r_N,e_{j_N})\big)$, $r=(r_1,\dots,r_N)$, $dr=dr_1\cdots dr_N$. Then, by (7.1) and (7.10), we have that

$$\begin{aligned}
E\Big(\big\|D^N u^n(t,x)\big\|^p_{\mathcal{H}_T^{\otimes N}}\Big)&=E\left(\int_{[0,T]^N}dr\sum_{j_1,\dots,j_N}\big|D^N_\alpha u^n(t,x)\big|^2\right)^{\frac{p}{2}}\\
&\le C\sum_{i=1}^5 N_i,
\end{aligned}$$

where

$$\begin{aligned}
N_1=E\Bigg(\int_{[0,T]^N}dr\sum_{j_1,\dots,j_N}\bigg|\sum_{i=1}^N\Big\langle\Lambda(t-r_i,x-*)\\
\times D^{N-1}_{\hat\alpha_i}\sigma\big(u^{n-1}(r_i,*)\big),e_{j_i}\Big\rangle_{\mathcal{H}}\bigg|^2\Bigg)^{\frac{p}{2}},
\end{aligned}$$

$$N_2 = E\Bigg(\int_{[0,T]^N} dr \sum_{j_1,\dots,j_N} \bigg|\int_{\bigvee_i r_i}^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z)$$

$$\times \triangle_\alpha^N\big(\sigma, u^{n-1}(s,z)\big) M(ds,dz)\bigg|^2\Bigg)^{\frac{p}{2}},$$

$$N_3 = E\Bigg(\int_{[0,T]^N} dr \sum_{j_1,\dots,j_N} \bigg|\int_{\bigvee_i r_i}^t ds \int_{\mathbb{R}^d} \Lambda(s,dz)$$

$$\times \triangle_\alpha^N\big(b, u^{n-1}(t-s,x-z)\big)\bigg|^2\Bigg)^{\frac{p}{2}},$$

$$N_4 = E\Bigg(\int_{[0,T]^N} dr \sum_{j_1,\dots,j_N} \bigg|\int_{\bigvee_i r_i}^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z) D_\alpha^N u^{n-1}(s,z)$$

$$\times \sigma'\big(u^{n-1}(s,z)\big) M(ds,dz)\bigg|^2\Bigg)^{\frac{p}{2}},$$

$$N_5 = E\Bigg(\int_{[0,T]^N} dr \sum_{j_1,\dots,j_N} \bigg|\int_{\bigvee_i r_i}^t ds \int_{\mathbb{R}^d} \Lambda(s,dz) D_\alpha^N u^{n-1}(t-s,x-z)$$

$$\times b'\big(u^{n-1}(t-s,x-z)\big)\bigg|^2\Bigg)^{\frac{p}{2}}.$$

By Parseval's identity and the definition of the $\mathcal{H}$-norm it follows that

$$N_1 \le C \sum_{i=1}^N E\Bigg(\int_{[0,T]^N} dr \sum_{j_1,\dots,j_N} \bigg|\Big\langle \Lambda(t-r_i, x-*)$$

$$\times D_{\hat{\alpha}_i}^{N-1} \sigma\big(u^{n-1}(r_i,*)\big), e_{j_i}\Big\rangle_{\mathcal{H}}\bigg|^2\Bigg)^{\frac{p}{2}}$$

$$= C \sum_{i=1}^n E\left(\int_{[0,T]^N} dr \sum_{\hat{j}_i} \Big\| \Lambda(t-r_i, x-*) D_{\hat{\alpha}_i}^{N-1} \sigma\big(u^{n-1}(r_i,*)\big)\Big\|_{\mathcal{H}}^2\right)^{\frac{p}{2}}$$

$$
\begin{aligned}
&= C\sum_{i=1}^{n} E\bigg(\int_{[0,T]^N} dr \int_{\mathbb{R}^d}\Gamma(dz)\int_{\mathbb{R}^d} dy \\
&\quad \times \Lambda(t-r_i,x-y)\Lambda(t-r_i,x-y+z) \\
&\quad \times \bigg[\sum_{\hat{j}_i} D^{N-1}_{\hat{\alpha}_i}\sigma\big(u^{n-1}(r_i,y)\big)D^{N-1}_{\hat{\alpha}_i}\sigma\big(u^{n-1}(r_i,y-z)\big)\bigg]\bigg)^{\frac{p}{2}},
\end{aligned}
$$

where $\hat{j}_i = j_1,\dots,j_{i-1},j_{i+1},\dots,j_N$. Then, by the Cauchy-Schwarz inequality and Hölder's inequality the preceding expression is bounded by

$$
\begin{aligned}
&\sum_{i=1}^{n} E\bigg(\int_0^T dr_i \int_{\mathbb{R}^d}\Gamma(dz)\int_{\mathbb{R}^d} dy\Lambda(t-r_i,x-y)\Lambda(t-r_i,x-y+z) \\
&\qquad \times \int_{[0,T]^{N-1}} d\hat{r}_i \Big\|D^{N-1}_{\hat{r}_i}\sigma\big(u^{n-1}(r_i,y)\big)\Big\|_{\mathcal{H}^{\otimes(N-1)}} \\
&\qquad \times \Big\|D^{N-1}_{\hat{r}_i}\sigma\big(u^{n-1}(r_i,y-z)\big)\Big\|_{\mathcal{H}^{\otimes(N-1)}}\bigg)^{\frac{p}{2}} \\
&\le C\sum_{i=1}^{n}\int_0^T dr_i \int_{\mathbb{R}^d}\Gamma(dz)\int_{\mathbb{R}^d} dy\Lambda(t-r_i,x-y)\Lambda(t-r_i,x-y+z) \\
&\qquad \times E\bigg(\int_{[0,T]^{N-1}} d\hat{r}_i \Big\|D^{N-1}_{\hat{r}_i}\sigma\big(u^{n-1}(r_i,y)\big)\Big\|_{\mathcal{H}^{\otimes(N-1)}} \\
&\qquad \times \Big\|D^{N-1}_{\hat{r}_i}\sigma\big(u^{n-1}(r_i,y-z)\big)\Big\|_{\mathcal{H}^{\otimes(N-1)}}\bigg)^{\frac{p}{2}} \\
&\le C\sum_{i=1}^{n}\int_0^T dr_i \int_{\mathbb{R}^d}\Gamma(dz)\int_{\mathbb{R}^d} dy\Lambda(t-r_i,x-y)\Lambda(t-r_i,x-y+z) \\
&\qquad \times \sup_{v\in\mathbb{R}^d} E\bigg(\int_{[0,T]^{N-1}} d\hat{r}_i \Big\|D^{N-1}_{\hat{r}_i}\sigma\big(u^{n-1}(r_i,v)\big)\Big\|^2_{\mathcal{H}^{\otimes(N-1)}}\bigg)^{\frac{p}{2}} \\
&\le C \sup_{(s,z)\in[0,T]\times\mathbb{R}^d} E\bigg(\Big\|D^{N-1}\sigma\big(u^{n-1}(s,z)\big)\Big\|^p_{\mathcal{H}_T^{\otimes(N-1)}}\bigg),
\end{aligned}
$$

with $d\hat{r}_i = dr_1\cdots dr_{i-1}\,dr_{i+1}\cdots dr_N$.

Then, by Leibniz's rule, the assumptions on σ and the induction hypothesis, it follows that N_1 is uniformly bounded with respect to n, t and x.

In the remaining terms we can replace $\bigvee_i r_i$ by 0, because the Malliavin derivatives involved vanish for $t < \bigvee_i r_i$.

By Theorem 6.1

$$\begin{aligned}
N_2 &= E\left(\left\|\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z) \triangle^N\big(\sigma, u^{n-1}(s,z)\big) M(ds,dz)\right\|^p_{\mathcal{H}_T^{\otimes N}}\right) \\
&\le C \int_0^t ds \sup_{y\in\mathbb{R}^d} E\left(\left\|\triangle^N\big(\sigma, u^{n-1}(s,y)\big)\right\|^p_{\mathcal{H}_T^{\otimes N}}\right) \int_{\mathbb{R}^d} \mu(d\xi)\left|\mathcal{F}\Lambda(t-s)(\xi)\right|^2 \\
&\le C \int_0^t ds \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} E\left(\left\|\triangle^N\big(\sigma, u^{n-1}(\tau,y)\big)\right\|^p_{\mathcal{H}_T^{\otimes N}}\right) J(t-s),
\end{aligned}$$

with $J(t) = \int_{\mathbb{R}^d} \mu(d\xi)\left|\mathcal{F}\Lambda(t)(\xi)\right|^2$. According to the induction hypothesis, this last term is uniformly bounded with respect to n, t and x.

Using similar arguments —this time for deterministic integration of Hilbert-valued processes— Hölder's inequality and the assumptions on Λ, we obtain

$$\begin{aligned}
N_3 &\le C \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz) E\left\|\triangle^N\big(b, u^{n-1}(t-s,x-z)\big)\right\|^p_{\mathcal{H}_T^{\otimes N}} \\
&\le C \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} E\left(\left\|\triangle^N\big(b, u^{n-1}(s,y)\big)\right\|^p_{\mathcal{H}_T^{\otimes N}}\right),
\end{aligned}$$

which again, by the induction hypothesis, is uniformly bounded in n, t and x.

For N_4 we proceed as for N_2; this yields,

$$N_4 \le C \int_0^t ds \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} E\left(\left\|D^N\big(u^{n-1}(\tau,y)\big)\right\|^p_{\mathcal{H}_T^{\otimes N}}\right) J(t-s).$$

Finally, as for N_3,

$$N_5 \le C \int_0^t ds \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} E\left(\left\|D^N u^{n-1}(\tau,y)\right\|^p_{\mathcal{H}_T^{\otimes N}}\right).$$

Summarising the estimates obtained so far we get

$$\begin{aligned}
&\sup_{(s,y)\in[0,t]\times\mathbb{R}^d} E\left(\left\|D^N u^n(s,y)\right\|^p_{\mathcal{H}_T^{\otimes N}}\right) \\
&\le C\left[1 + \int_0^t ds \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} E\left(\left\|D^N u^{n-1}(\tau,y)\right\|^p_{\mathcal{H}_T^{\otimes N}}\right)\big(J(t-s)+1\big)\right].
\end{aligned}$$

Then, the proof ends applying the version of Gronwall's lemma given in Lemma 6.2. □

Lemma 7.4

We suppose that the assumptions of Proposition 7.1 are satisfied. Then for any positive integer $N \geq 1$ the sequence $D^N u^n(t,x)$, $n \geq 0$, converges in $L^2(\Omega; \mathcal{H}_T^{\otimes N})$, uniformly in $(t,x) \in [0,T] \times \mathbb{R}^d$, to the $\mathcal{H}_T^{\otimes N}$-valued stochastic processes $\big(u(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\big)$ solution of the equation

$$\begin{aligned} U(t,x) = Z^N(t,x) &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z) \Big[\triangle^N\big(\sigma, u(s,z)\big) \\ &\quad + U(s,z)\sigma'\big(u(s,z)\big)\Big] M(ds,dz) \\ &+ \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz) \Big[\triangle^N\big(b, u(t-s,x-z)\big) \\ &\quad + U(t-s,x-z) b'\big(u(t-s,x-z)\big)\Big], \end{aligned} \tag{7.12}$$

with Z^N given by (7.6).

Proof

Here again we use induction on N. For $N = 1$ we proceed as follows. Let $\big(U(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\big)$ be the solution of

$$\begin{aligned} U(t,x) = \Lambda(t-\cdot, x-*)\sigma\big(u(\cdot,*)\big) &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z) \\ &\times \sigma'\big(u(s,z)\big) U(s,z) M(ds,dz) \\ &+ \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz) b'\big(u(t-s,x-z)\big) U(t-s,x-z). \end{aligned} \tag{7.13}$$

We prove that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(\big\|Du^n(t,x) - U(t,x)\big\|_{\mathcal{H}_T}^2\Big) \longrightarrow 0, \tag{7.14}$$

as n tends to infinity. This implies that $u(t,x) \in \mathbb{D}^{1,2}$ and the process $\big\{Du(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\big\}$ satisfies equation (7.13).

Set

$$I_Z^n(t,x) = \Lambda(t-\cdot, x-*)\Big(\sigma\big(u^{n-1}(\cdot,*)\big) - \sigma\big(u(\cdot,*)\big)\Big),$$

$$\begin{aligned} I_\sigma^n(t,x) = &\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z)\sigma'\big(u^{n-1}(s,z)\big) Du^{n-1}(s,z) M(ds,dz) \\ &- \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z)\sigma'\big(u(s,z)\big) U(s,z) M(ds,dz), \end{aligned}$$

$$I_b^n(t,x) = \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz)\Big(b'\big(u^{n-1}(t-s,x-z)\big)Du^{n-1}(t-s,x-z) \\ - b'\big(u(t-s,x-z)\big)U(t-s,x-z)\Big).$$

The Lipschitz property of σ yields

$$E\Big(\big\|I_Z^n(t,x)\big\|_{\mathcal{H}_T}^2\Big) \le C \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(\big|u^{n-1}(t,x)-u(t,x)\big|^2\Big) \\ \times \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\big|\mathcal{F}\Lambda(s)(\xi)\big|^2 \\ \le C \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(\big|u^{n-1}(t,x)-u(t,x)\big|^2\Big).$$

Hence,

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(\big\|I_Z^n(t,x)\big\|_{\mathcal{H}_T}^2\Big) = 0. \tag{7.15}$$

Consider the decomposition

$$E\Big(\big\|I_\sigma^n(t,x)\big\|_{\mathcal{H}_T}^2\Big) \le C(D_{1,n}(t,x) + D_{2,n}(t,x),$$

where

$$D_{1,n}(t,x) = E\bigg(\bigg\|\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z)\Big[\sigma'\big(u^{n-1}(s,z)\big) \\ - \sigma'\big(u(s,z)\big)\Big] Du^{n-1}(s,z)M(ds,dz)\bigg\|_{\mathcal{H}_T}^2\bigg),$$

$$D_{2,n}(t,x) = E\bigg(\bigg\|\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z)\sigma'\big(u(s,z)\big)\Big[Du^{n-1}(s,z) \\ - U(s,z)\Big]M(ds,dz)\bigg\|_{\mathcal{H}_T}^2\bigg).$$

The isometry property of the stochastic integral, Cauchy-Schwarz's inequality and the properties of σ yield

$$D_{1,n}(t,x) \le C \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \bigg(E\Big(\big|u^{n-1}(s,y)-u(s,y)\big|^4\Big) \\ \times E\Big(\big\|Du^{n-1}(s,y)\big\|_{\mathcal{H}_T}^4\Big)\bigg)^{\frac{1}{2}} \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\big|\mathcal{F}\Lambda(s)(\xi)\big|^2.$$

Owing to Theorem 6.2 and Lemma 7.3 we conclude that

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} D_{1,n}(t,x) = 0.$$

Similarly,

$$D_{2,n}(t,x) \le C \int_0^t ds \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} E\Big(\big\|Du^{n-1}(\tau,y) - U(\tau,y)\big\|^2_{\mathcal{H}_T}\Big) J(t-s). \tag{7.16}$$

For the pathwise integral term, we have

$$E\Big(\big\|I_b^n(t,x)\big\|^2_{\mathcal{H}_T}\Big) \le C\big(b_{1,n}(t,x) + b_{2,n}(t,x)\big),$$

with

$$\begin{aligned} b_{1,n}(t,x) = E\Bigg(\bigg\|\int_0^t ds \int_{\mathbb{R}^d} &\Lambda(s,dz) \\ &\times \Big[b'\big(u^{n-1}(t-s,x-z)\big) - b'\big(u(t-s,x-z)\big)\Big] \\ &\times Du^{n-1}(t-s,x-z)\bigg\|^2_{\mathcal{H}_T}\Bigg), \\ b_{2,n}(t,x) = E\Bigg(\bigg\|\int_0^t ds \int_{\mathbb{R}^d} &\Lambda(s,dz) b'\big(u(t-s,x-z)\big) \\ &\times \Big[Du^{n-1}(t-s,x-z) - U(t-s,x-z)\Big]\bigg\|^2_{\mathcal{H}_T}\Bigg). \end{aligned}$$

By the properties of the deterministic integral of Hilbert-valued processes, the assumptions on b, and Cauchy-Schwarz's inequality we obtain

$$\begin{aligned} b_{1,n}(t,x) &\le \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz) \\ &\qquad \times E\bigg(\Big|b'\big(u^{n-1}(t-s,x-z)\big) - b'\big(u(t-s,x-z)\big)\Big|^2 \\ &\qquad \times \big\|Du^{n-1}(t-s,x-z)\big\|^2_{\mathcal{H}_T}\bigg) \\ &\le \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \Big(E\big|u^{n-1}(s,y) - u(s,y)\big|^4 E\big\|Du^{n-1}(s,y)\big\|^4_{\mathcal{H}_T}\Big)^{\frac{1}{2}} \\ &\qquad \times \int_0^t ds\Lambda(s,dz). \end{aligned}$$

Thus, $\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} b_{1,n}(t,x) = 0$.

Similar arguments yield

$$b_{2,n}(t,x) \le C \int_0^t ds \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} E\Big(\big\|Du^{n-1}(\tau,y) - U(\tau,y)\big\|^2_{\mathcal{H}_T}\Big).$$

Therefore we have obtained that

$$\begin{aligned}
&\sup_{(s,x)\in[0,t]\times\mathbb{R}^d} E\Big(\big\|Du^n(s,x) - U(s,x)\big\|^2_{\mathcal{H}_T}\Big)\\
&\quad \le C_n + C\int_0^t ds \sup_{(\tau,x)\in[0,s]\times\mathbb{R}^d} E\Big(\big\|Du^{n-1}(\tau,x)\\
&\qquad - U(\tau,x)\big\|^2_{\mathcal{H}_T}\Big)\big(J(t-s)+1\big),
\end{aligned}$$

with $\lim_{n\to\infty} C_n = 0$. Thus applying Gronwall's Lemma 6.2 we complete the proof of (7.14).

We now assume that the convergence in quadratic mean holds for all derivatives up to the order $N-1$, uniformly in $(t,x)\in[0,T]\times\mathbb{R}^d$, and prove the same result for the order N. That means we must check that

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(\big\|D^N u^n(t,x) - U(t,x)\big\|^2_{\mathcal{H}_T^{\otimes N}}\Big) = 0, \tag{7.17}$$

where $D^N u^n(t,x)$, $U(t,x)$ satisfy the equations (7.10), (7.12), respectively.

We start with the convergence of the terms playing the role of initial conditions.

Set

$$\begin{aligned}
Z^n := E\Bigg(\int_{[0,T]^N} dr \sum_{j_1,\dots,j_N} \Bigg|\sum_{i=1}^N \Big\langle \Lambda(t-r_i, x-*)\\
\times \Big[D^{N-1}_{\hat\alpha_i}\sigma\big(u^{n-1}(r_i,*)\big) - D^{N-1}_{\hat\alpha_i}\sigma\big(u(r_i,*)\big)\Big], e_{j_i}\Big\rangle_{\mathcal{H}}\Bigg|^2\Bigg).
\end{aligned}$$

Then, Parseval's identity and Cauchy-Schwarz inequality ensure

$$\begin{aligned}
Z^n = \sum_{i=1}^N E\Bigg(\int_{[0,T]^{N-1}} d\hat r_i\\
\times \sum_{\hat j_i} \Big\|\Lambda(t-\cdot, x-*)\Big[D^{N-1}_{\hat\alpha_i}\sigma\big(u^{n-1}(\cdot,*)\big) - D^{N-1}_{\hat\alpha_i}\sigma\big(u(\cdot,*)\big)\Big]\Big\|^2_{\mathcal{H}_T}\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{N} E\bigg(\int_{[0,T]^{N-1}} d\hat{r}_i \int_0^t ds \int_{\mathbb{R}^d} \Gamma(dz) \int_{\mathbb{R}^d} dy \Lambda(t-s, x-y) \\
&\quad \times \Lambda(t-s, x-y+z) \Big\| D_{\hat{r}_i}^{N-1}\Big(\sigma\big(u^{n-1}(s,y)\big) - \sigma\big(u(s,y)\big)\Big)\Big\|_{\mathcal{H}^{\otimes(N-1)}} \\
&\quad \times \Big\| D_{\hat{r}_i}^{N-1}\Big(\sigma\big(u^{n-1}(s,y-z)\big) - \sigma\big(u(s,y-z)\big)\Big)\Big\|_{\mathcal{H}^{\otimes(N-1)}}\bigg) \\
&\leq \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} E\bigg(\Big\| D^{N-1}\Big(\sigma\big(u^{n-1}(s,y)\big) - \sigma\big(u(s,y)\big)\Big)\Big\|^2_{\mathcal{H}_T^{\otimes(N-1)}}\bigg) \\
&\quad \times \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) \big|\mathcal{F}\Lambda(t-s)(\xi)\big|^2 \\
&\leq C \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} E\bigg(\Big\| D^{N-1}\Big(\sigma\big(u^{n-1}(s,y)\big) - \sigma\big(u(s,y)\big)\Big)\Big\|^2_{\mathcal{H}_T^{\otimes(N-1)}}\bigg).
\end{aligned}
$$

Leibniz's formula, the result proved in Lemma 7.3, and the induction assumption yield that te last term tends to zero as n goes to infinity.

We finish the proof by similar arguments as those used for $N = 1$. We omit the details. □

We are now prepared to give the proof of Proposition 7.1.

Proof of Proposition 7.1

We apply Lemma 7.1 to the sequence of random variables consisting of the Picard iterations for the process u defined in (7.7). More precisely, fix $(t,x) \in [0,T] \times \mathbb{R}^d$ and set $F_n = u^n(t,x)$, $F = u(t,x)$.

By Theorem 13 of reference [14] (see also Theorem 6.2),

$$
\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(\big|u^n(t,x) - u(t,x)\big|^p\Big) = 0.
$$

Thus, assumption (a) in Lemma 7.1 —which does not need any kind of differentiability— is satisfied. The validity of assumption (b) follows from Lemmas 7.2 to 7.4 above. □

Remark 7.1 The absolute continuity of $\Lambda(t)$ is only used in the analysis of the terms involving σ but not in those involving b.

We now shall deal with more general Λ. Let ψ be a $\mathcal{C}^\infty(\mathbb{R}^d)$ function with compact support contained in the unit ball of $\mathbb{R}^d$. Define $\psi_n(x) = n^d\psi(nx)$ and

$$\Lambda_n(t) = \psi_n * \Lambda(t), \tag{7.18}$$

$n \geq 1$. It is well known that $\Lambda_n(t)$ is a $\mathcal{C}^\infty(\mathbb{R}^d)$ function.

Moreover,

$$\left|\mathcal{F}\Lambda_n(t)\right| \leq \left|\mathcal{F}\Lambda(t)\right|. \tag{7.19}$$

Consider the sequence of processes $\big(u_n(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\big)$ solution to the equations

$$\begin{aligned} u_n(t,x) = &\int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s, x-z)\sigma\big(u_n(s,z)\big)M(ds,dz) \\ &+ \int_0^t ds \int_{\mathbb{R}^d} b\big(u_n(t-s,x-z)\big)\Lambda(s,dz). \end{aligned} \tag{7.20}$$

Under the assumptions of Theorem 7.1 we conclude from Proposition 7.1 and Remark 7.1 that $u_n(t,x) \in \mathbb{D}^\infty$, for all $n \geq 1$.

Moreover, the derivative $D^N u_n(t,x)$ satisfies the equation

$$\begin{aligned} D_\alpha^N u_n(t,x) = &\sum_{i=1}^N \Big\langle \Lambda_n(t-r_i, x-*) D_{\hat{\alpha}_i}^{N-1} \sigma\big(u_n(r_i,*)\big), \varphi_i \Big\rangle_{\mathcal{H}} \\ &+ \int_{\bigvee_i r_i}^t \int_{\mathbb{R}^d} \Lambda_n(t-s,x-z) \Big[\triangle_\alpha^N\big(\sigma, u_n(s,z)\big) \\ &\qquad + D_\alpha^N u_n(s,z)\sigma'\big(u_n(s,z)\big)\Big] M(ds,dz) \\ &+ \int_{\bigvee_i r_i}^t ds \int_{\mathbb{R}^d} \Lambda(s,dz) \Big[\triangle_\alpha^N\big(b, u_n(t-s,x-z)\big) \\ &\qquad + D_\alpha^N u_n(t-s,x-z) b'\big(u_n(t-s,x-z)\big)\Big], \end{aligned} \tag{7.21}$$

where $\alpha = \big((r_1,\varphi_1),\dots,(r_N,\varphi_N)\big)$, with $r_1,\dots,r_N \geq 0$ and $\varphi_1,\dots,\varphi_N \in \mathcal{H}$. With these tools we can now give the ingredients for the proof of Theorem 7.1.

Lemma 7.5

Assume that the coefficients σ and b are Lipschitz continuous and that Hypothesis D is satisfied. Then for any $p \in [1,\infty)$,

$$\lim_{n\to\infty} \left(\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(\big|u_n(t,x) - u(t,x)\big|^p\Big)\right) = 0. \tag{7.22}$$

PROOF

We first prove that for any $p \in [1,\infty)$,

$$\sup_{n\geq 1} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(\big|u_n(t,x)\big|^p\Big) < \infty. \tag{7.23}$$

Taking into account (7.20), we have $E\big(|u_n(t,x)|^p\big) \leq C\big(A_{1,n}(t,x) + A_{2,n}(t,x)\big)$, where

$$A_{1,n}(t,x) = E\left(\left|\int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s,x-z)\sigma\big(u_n(s,z)\big)M(ds,dz)\right|^p\right),$$
$$A_{2,n}(t,x) = E\left(\left|\int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz)b\big(u_n(t-s,x-z)\big)\right|^p\right).$$

Owing to Theorem 6.1, the properties of σ, the definition of Λ_n and (7.19), we obtain

$$\begin{aligned} &A_{1,n}(t,x) \\ &\quad\leq C\nu(t)^{\frac{p}{2}-1}\int_0^t ds \sup_{z\in\mathbb{R}^d} E\Big(\big|\sigma\big(u_n(s,z)\big)\big|^p\Big)\int_{\mathbb{R}^d}\mu(d\xi)\big|\mathcal{F}\Lambda(t-s)(\xi)\big|^2 \\ &\quad\leq C\int_0^t ds\left(1+\sup_{z\in\mathbb{R}^d} E\Big(\big|u_n(s,z)\big|^p\Big)\right)\int_{\mathbb{R}^d}\mu(d\xi)\big|\mathcal{F}\Lambda(t-s)(\xi)\big|^2, \end{aligned}$$

with $\nu(t) = \int_0^t ds \int_{\mathbb{R}^d}\mu(d\xi)\big|\mathcal{F}\Lambda(s)(\xi)\big|^2$.

Consequently,

$$A_{1,n}(t,x) \leq C\int_0^t ds\left[1+\sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} E\Big(\big|u_n(\tau,y)\big|^p\Big)\right]J(t-s). \tag{7.24}$$

Hölder's inequality with respect to the finite measure $\Lambda(s,dz)ds$, the properties of b, and Hypothesis D yield

$$\begin{aligned} A_{2,n}(t,x) &\leq C\int_0^t ds\int_{\mathbb{R}^d}\Lambda(s,dz)E\Big(\big|b\big(u_n(t-s,x-z)\big)\big|^p\Big) \\ &\leq C\int_0^t ds\left[1+\sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} E\Big(\big|u_n(\tau,y)\big|^p\Big)\right]\int_{\mathbb{R}^d}\Lambda(t-s,dz) \\ &\leq C\int_0^t ds\left[1+\sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} E\Big(\big|u_n(\tau,y)\big|^p\Big)\right]. \end{aligned} \tag{7.25}$$

Putting together (7.24) and (7.25) we obtain

$$\sup_{(s,x)\in[0,t]\times\mathbb{R}^d} E\Big(\big|u_n(s,x)\big|^p\Big) \le C\int_0^t ds\left[1+\sup_{(\tau,x)\in[0,s]\times\mathbb{R}^d} E\Big(\big|u_n(\tau,x)\big|^p\Big)\right]\big(J(t-s)+1\big).$$

Then we apply the version of Gronwall's Lemma given in Lemma 6.2 and finish the proof of (7.23).

Next we prove that

$$\lim_{n\to\infty}\left(\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(\big|u_n(t,x)-u(t,x)\big|^2\Big)\right)=0. \tag{7.26}$$

Indeed, according to the integral equations (7.20) and (6.19), we have

$$E\Big(\big|u_n(t,x)-u(t,x)\big|^2\Big) \le C\big(I_{1,n}(t,x)+I_{2,n}(t,x)\big),$$

where

$$I_{1,n}(t,x) = E\bigg(\Big|\int_0^t\int_{\mathbb{R}^d}\Big[\Lambda_n(t-s,x-z)\sigma\big(u_n(s,z)\big) - \Lambda(t-s,x-z)\sigma\big(u(s,z)\big)\Big]M(ds,dz)\Big|^2\bigg),$$

$$I_{2,n}(t,x) = E\bigg(\Big|\int_0^t ds\int_{\mathbb{R}^d}\Lambda(s,dz)\Big[b\big(u_n(t-s,x-z)\big) - b\big(u(t-s,x-z)\big)\Big]\Big|^2\bigg).$$

We have $I_{1,n}(t,x)\le C\big(I_{1,n}^1(t,x)+I_{1,n}^2(t,x)\big)$ with

$$I_{1,n}^1(t,x) = E\left(\Big|\int_0^t\int_{\mathbb{R}^d}\Lambda_n(t-s,x-z) \times\Big[\sigma\big(u_n(s,z)\big)-\sigma\big(u(s,z)\big)\Big]M(ds,dz)\Big|^2\right),$$

$$I_{1,n}^2(t,x) = E\left(\Big|\int_0^t\int_{\mathbb{R}^d}\big[\Lambda_n(t-s,x-z) - \Lambda(t-s,x-z)\big]\sigma\big(u(s,z)\big)M(ds,dz)\Big|^2\right).$$

By the isometry property of the stochastic integral, the assumptions on σ and the definition of Λ_n, we obtain

$$I_{1,n}^1(t,x) \le C \int_0^t ds \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} E\Big(\big|u_n(\tau,y)-u(\tau,y)\big|^2\Big) J(t-s).$$

Although $\Lambda_n(t-s)-\Lambda(t-s)$ may not be a non-negative distribution, it does belong to the space $\mathcal{P}_{0,\sigma(u)}$ of deterministic processes integrable with respect to the martingale measure $M^{\sigma(u)}$. Hence, by the isometry property of the stochastic integral,

$$I_{1,n}^2(t,x) = \big\|\Lambda_n(t-\cdot,x-*)-\Lambda(t-\cdot,x-*)\big\|_{0,\sigma(u)}^2.$$

Then, the definition of the norm in the right-hand side of the above equality yields

$$\begin{aligned} I_{1,n}^2(t,x) &= \int_0^t ds \int_{\mathbb{R}^d} \mu_s^{\sigma(u)}(d\xi)\Big|\mathcal{F}\big(\Lambda_n(t-s)-\Lambda(t-s)\big)(\xi)\Big|^2 \\ &= \int_0^t ds \int_{\mathbb{R}^d} \mu_s^{\sigma(u)}(d\xi)\big|\mathcal{F}\psi_n(\xi)-1\big|^2 \big|\mathcal{F}\Lambda(t-s)(\xi)\big|^2. \end{aligned}$$

Hence, by bounded convergence we conclude that

$$C_n := \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} I_{1,n}^2(t,x)$$

tends to zero as n goes to infinity.

Now we study the term $I_{2,n}(t,x)$. Applying the same techniques as for the term $A_{2,n}(t,x)$ before we obtain

$$I_{2,n}(t,x) \le C \int_0^t ds \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} E\Big(\big|u_n(\tau,y)-u(\tau,y)\big|^2\Big).$$

Consequently,

$$\begin{aligned} &\sup_{(s,x)\in[0,t]\times\mathbb{R}^d} E\Big(\big|u_n(s,x)-u(s,x)\big|^2\Big) \\ &\quad \le C_n + C\int_0^t ds \sup_{(\tau,x)\in[0,s]\times\mathbb{R}^d} E\Big(\big|u_n(\tau,x)-u(\tau,x)\big|^2\Big)\big(J(t-s)+1\big), \end{aligned}$$

where $\lim_{n\to\infty} C_n = 0$.

The proof of (7.26) concludes with an application of the above mentioned version of Gronwall's Lemma.

The convergence (7.22) is now a consequence of (7.23) and (7.26). □

Lemma 7.6

Assume that the assumptions of Theorem 7.1 are satisfied. Then, for any positive integer $N \geq 1$ and $p \in [1, \infty)$,

$$\sup_{n \geq 0} \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} E\Big(\|D^N u_n(t,x)\|^p_{\mathcal{H}_T^{\otimes N}} \Big) < \infty. \tag{7.27}$$

PROOF

We follow exactly the same scheme as in the proof of Lemma 7.3, with the obvious changes. In the estimates we must use the above mentioned property (7.19). We omit the details. □

The next result is a step further towards the identification of the stochastic process $Z^N(t,x)$ appearing in the right hand-side of (7.3).

For $N \geq 1$, $n \geq 1$, $r = (r_1, \dots, r_N)$, $\alpha = \big((r_1, e_{j_1}), \dots, (r_N, e_{j_N})\big)$ and $(t,x) \in [0,t] \times \mathbb{R}^d$, we define the $\mathcal{H}^{\otimes N}$-valued random variable $Z_r^{N,n}(t,x)$ as follows,

$$\big\langle Z_r^{N,n}(t,x), e_{j_1} \otimes \cdots \otimes e_{j_N} \big\rangle_{\mathcal{H}^{\otimes N}} = \sum_{i=1}^N \Big\langle \Lambda_n(t - r_i, x - *) D^{N-1}_{\hat{\alpha}_i} \sigma\big(u_n(r_i, *)\big),\, e_{j_i} \Big\rangle_{\mathcal{H}}.$$

Applying Lemma 7.6 it can be easily seen that $Z^{N,n}(t,x) \in L^p(\Omega; \mathcal{H}_T^{\otimes N})$, and

$$\sup_{n \geq 1} \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} E\Big(\|Z^{N,n}(t,x)\|^p_{\mathcal{H}_T^{\otimes N}} \Big) < +\infty, \tag{7.28}$$

for every $p \in [1, \infty)$.

Notice that $Z^{N,n}(t,x)$ coincides with the first term of the right hand-side of Equation (7.21) for $\alpha = \big((r_1, e_{j_1}), \dots, (r_N, e_{j_N})\big)$.

For $N \geq 1$ we introduce the following assumption:

Assumption ($\mathbf{H}_{N-1}$) *The sequence $\{D^j u_n(t,x), n \geq 1\}$ converges in $L^p(\Omega; \mathcal{H}_T^{\otimes j})$, $j = 1, \dots, N-1$, with the convention that $L^p(\Omega; \mathcal{H}_T^{\otimes 0}) = L^p(\Omega)$.*

Lemma 7.5 yields the validity of (H_0). Moreover, for $N > 1$, (H_{N-1}) implies that $u(t,x) \in \mathbb{D}^{j,p}$ and the sequences $\{D^j u_n(t,x), n \geq 1\}$ converge in $L^p(\Omega; \mathcal{H}_T^{\otimes j})$ to $D^j u(t,x)$. In addition, by Lemma 7.6

$$\sup_{(s,y) \in [0,T] \times \mathbb{R}^d} E\Big(\|D^j u(s,y)\|^p_{\mathcal{H}_T^{\otimes j}} \Big) < \infty, \qquad j = 1, \dots, N-1. \tag{7.29}$$

Lemma 7.7

Fix $N \geq 1$. Assume the same hypothesis as in Theorem 7.1 and that (H_{N-1}) holds. Then the sequence $\{Z^{N,n}(t,x)\}_{n\geq 1}$ converges in $L^p(\Omega;\mathcal{H}_T^{\otimes N})$ to a random variable $Z^N(t,x)$.

Proof

Consider first the case $N=1$. Then

$$Z^{1,n}(t,x) = \Lambda_n(t-\cdot,x-*)\sigma\big(u_n(\cdot,*)\big).$$

We prove that $\{Z^{1,n}(t,x), n\geq 1\}$ is a Cauchy sequence in $L^2(\Omega;\mathcal{H}_T)$. Indeed, for any $n,m\geq 1$ we consider the following decomposition:

$$E\left(\Big\|\Lambda_n(t-\cdot,x-*)\sigma\big(u_n(\cdot,*)\big) - \Lambda_m(t-\cdot,x-*)\sigma\big(u_m(\cdot,*)\big)\Big\|_{\mathcal{H}_T}^2\right) \\ \leq C\big(T_{1,n}(t,x) + T_{2,n,m}(t,x) + T_{3,m}(t,x)\big),$$

where

$$T_{1,n}(t,x) = E\left(\Big\|\Lambda_n(t-\cdot,x-*)\Big[\sigma\big(u_n(\cdot,*)\big) - \sigma\big(u(\cdot,*)\big)\Big]\Big\|_{\mathcal{H}_T}^2\right),$$
$$T_{2,n,m}(t,x) = E\left(\Big\|\big[\Lambda_n(t-\cdot,x-*) - \Lambda_m(t-\cdot,x-*)\big]\sigma\big(u(\cdot,*)\big)\Big\|_{\mathcal{H}_T}^2\right),$$
$$T_{3,m}(t,x) = E\left(\Big\|\Lambda_m(t-\cdot,x-*)\Big[\sigma\big(u(\cdot,*)\big) - \sigma\big(u_m(\cdot,*)\big)\Big]\Big\|_{\mathcal{H}_T}^2\right).$$

Since Λ_n is a smooth function, the Lipschitz property of σ and the definition of Λ_n yield

$$T_{1,n}(t,x) \leq C \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} E\Big(\big|u_n(s,y) - u(s,y)\big|^2\Big) \\ \times \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\big|\mathcal{F}\Lambda_n(s)(\xi)\big|^2 \\ \leq C \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} E\Big(\big|u_n(s,y) - u(s,y)\big|^2\Big).$$

Then, by Lemma 7.5 we conclude that

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} T_{1,n}(t,x) = 0.$$

Similarly,

$$\lim_{m\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} T_{3,m}(t,x) = 0.$$

Owing to the isometry property of the stochastic integral we have

$$\begin{aligned} T_{2,n,m}(t,x) &= E\Big(\big\|\Lambda_n(t-\cdot,x-*)-\Lambda_m(t-\cdot,x-*)\big\|^2_{0,\sigma(u)}\Big)\\ &= \int_0^T ds\int_{\mathbb{R}^d}\mu_s^{\sigma(u)}(d\xi)\big|\mathcal{F}(\psi_n-\psi_m)(\xi)\big|^2\,\big|\mathcal{F}\Lambda(s)(\xi)\big|^2. \end{aligned}$$

Then, by dominated convergence we conclude that

$$\lim_{n,m\to\infty}\ \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} T_{2,n,m}(t,x)=0.$$

Therefore,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\bigg(\Big\|\Lambda_n(t-\cdot,x-*)\sigma\big(u_n(\cdot,*)\big) -\Lambda_m(t-\cdot,x-*)\sigma\big(u_m(\cdot,*)\big)\Big\|^2_{\mathcal{H}_T}\bigg)$$

tends to zero as n,m tends to infinity and consequently the sequence $\big\{Z^{1,n}(t,x),n\geq 1\big\}$ converges in $L^2(\Omega;\mathcal{H}_T)$ to a random variable denoted by $Z(t,x)$.

Actually, the convergence holds in $L^p(\Omega,\mathcal{H}_T)$ for any $p\in[1,\infty)$. Indeed

$$\sup_{n\geq 1}\ \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\bigg(\Big\|\Lambda_n(t-\cdot,x-*)\sigma\big(u_n(\cdot,*)\big)\Big\|^p_{\mathcal{H}_T}\bigg)<\infty.$$

This finishes the proof for $N=1$.

Assume $N>1$. In view of (7.28) it suffices to show that $\big\{Z^{N,n}(t,x)\big\}_{n\geq 1}$ is a Cauchy sequence in $L^2(\Omega;\mathcal{H}_T^{\otimes N})$.

For $n,m\geq 1$, set

$$\begin{aligned} Z^{n,m} := E\int_{[0,T]^N} dr \sum_{j_1,\dots,j_N}\bigg|&\sum_{i=1}^N\Big\langle \Lambda_n(t-r_i,x-*)D^{N-1}_{\hat{\alpha}_i}\sigma\big(u_n(r_i,*)\big),e_{j_i}\Big\rangle_{\mathcal{H}}\\ &-\sum_{i=1}^N\Big\langle \Lambda_m(t-r_i,x-*)D^{N-1}_{\hat{\alpha}_i}\sigma\big(u_m(r_i,*)\big),e_{j_i}\Big\rangle_{\mathcal{H}}\bigg|^2 \end{aligned}$$

Then,

$$Z^{n,m}\leq C(Z_1^n+Z_2^{n,m}+Z_3^m),$$

where

$$
\begin{aligned}
Z_1^n &= \sum_{i=1}^N E \int_{[0,T]^N} dr \sum_{j_1,\dots,j_N} \Big| \Big\langle \Lambda_n(t-r_i, x-*) \\
&\qquad \times \Big[D_{\hat{\alpha}_i}^{N-1} \sigma\big(u_n(r_i,*)\big) - D_{\hat{\alpha}_i}^{N-1} \sigma\big(u(r_i,*)\big) \Big], e_{j_i} \Big\rangle_{\mathcal{H}} \Big|^2, \\
Z_2^{n,m} &= \sum_{i=1}^N E \int_{[0,T]^N} dr \sum_{j_1,\dots,j_N} \Big| \Big\langle D_{\hat{\alpha}_i}^{N-1} \sigma\big(u(r_i,*)\big) \\
&\qquad \times \big[\Lambda_n(t-r_i, x-*) - \Lambda_m(t-r_i, x-*) \big], e_{j_i} \Big\rangle_{\mathcal{H}} \Big|^2, \\
Z_3^m &= \sum_{i=1}^N E \int_{[0,T]^N} dr \sum_{j_1,\dots,j_N} \Big| \Big\langle \Lambda_m(t-r_i, x-*) \\
&\qquad \times \Big[D_{\hat{\alpha}_i}^{N-1} \sigma\big(u(r_i,*)\big) - D_{\hat{\alpha}_i}^{N-1} \sigma\big(u_m(r_i,*)\big) \Big], e_{j_i} \Big\rangle_{\mathcal{H}} \Big|^2.
\end{aligned}
$$

Parseval's identity and the Cauchy-Schwarz inequality ensure

$$
\begin{aligned}
Z_1^n &= \sum_{i=1}^N E \int_{[0,T]^{N-1}} d\hat{r}_i \sum_{\hat{j}_i} \Big\| \Lambda_n(t-\cdot, x-*) \\
&\quad \times \Big[D_{\hat{\alpha}_i}^{N-1} \sigma\big(u_n(\cdot,*)\big) - D_{\hat{\alpha}_i}^{N-1} \sigma\big(u(\cdot,*)\big) \Big] \Big\|_{\mathcal{H}_T}^2 \\
&\le \sum_{i=1}^N E \int_{[0,T]^{N-1}} d\hat{r}_i \int_0^t ds \int_{\mathbb{R}^d} \Gamma(dz) \int_{\mathbb{R}^d} dy \Lambda_n(t-s, x-y) \\
&\quad \times \Lambda_n(t-s, x-y+z) \Big\| D_{\hat{r}_i}^{N-1} \Big(\sigma\big(u_n(s,y)\big) - \sigma\big(u(s,y)\big) \Big) \Big\|_{\mathcal{H}^{\otimes(N-1)}} \\
&\quad \times \Big\| D_{\hat{r}_i}^{N-1} \Big(\sigma\big(u_n(s,y-z)\big) - \sigma\big(u(s,y-z)\big) \Big) \Big\|_{\mathcal{H}^{\otimes(N-1)}} \\
&\le \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} E\bigg(\Big\| D^{N-1} \Big(\sigma\big(u_n(s,y)\big) - \sigma\big(u(s,y)\big) \Big) \Big\|_{\mathcal{H}_T^{\otimes(N-1)}}^2 \bigg) \\
&\quad \times \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) \big| \mathcal{F}\Lambda(t-s)(\xi) \big|^2 \\
&\le C \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} E\bigg(\Big\| D^{N-1} \Big(\sigma\big(u_n(s,y)\big) - \sigma\big(u(s,y)\big) \Big) \Big\|_{\mathcal{H}_T^{\otimes(N-1)}}^2 \bigg).
\end{aligned}
$$

Leibniz's rule, Lemma 7.6 and the assumption (H_{N-1}) yield that the last term tends to zero as n goes to infinity.

Analogously, Z_3^m tends to zero as m tends to infinity.

Using similar arguments we obtain

$$\begin{aligned}
Z_2^{n,m} &= \sum_{i=1}^{N} E \int_{[0,T]^{N-1}} d\hat{r}_i \\
&\quad \times \sum_{\hat{j}_i} \Big\| D^{N-1}_{\hat{\alpha}_i} \sigma\big(u(\cdot,*)\big) \big[\Lambda_n(t-\cdot, x-*) - \Lambda_m(t-\cdot, x-*)\big] \Big\|^2_{\mathcal{H}_T} \\
&= \sum_{i=1}^{N} E \int_{[0,T]^{N-1}} d\hat{r}_i \sum_{\hat{j}_i} \int_0^T ds \int_{\mathbb{R}^d} \Gamma(dz) \int_{\mathbb{R}^d} dy D^{N-1}_{\hat{\alpha}_i} \sigma\big(u(s,y)\big) \\
&\quad \times D^{N-1}_{\hat{\alpha}_i} \sigma\big(u(s,y-z)\big) \big[\Lambda_n(t-s, x-y) - \Lambda_m(t-s, x-y)\big] \\
&\quad \times \big[\Lambda_n(t-s, x-y+z) - \Lambda_m(t-s, x-y+z)\big] \\
&= \sum_{i=1}^{N} E \int_{[0,T]^{N-1}} d\hat{r}_i \sum_{\hat{j}_i} \int_0^T ds \int_{\mathbb{R}^d} \mu_s^{D^{N-1}_{\hat{\alpha}_i} \sigma(u)}(d\xi) \\
&\quad \times \Big| \mathcal{F}\big(\Lambda_n(t-s) - \Lambda_m(t-s)\big)(\xi) \Big|^2 .
\end{aligned}$$

This term tends to zero as m and n go to infinity. Indeed, arguing as in the proof of Theorem 2 from reference [14] (see also Theorem 6.1) we have that

$$\big\|\Lambda(t-\cdot)\big\|^2_{0,D^{N-1}_{\hat{\alpha}_i}\sigma(u)} \le \liminf_{k\to\infty} \big\|\Lambda_k(t-\cdot)\big\|^2_{0,D^{N-1}_{\hat{\alpha}_i}\sigma(u)} .$$

Then, by Fatou's Lemma

$$\begin{aligned}
E \int_{[0,T]^{N-1}} d\hat{r}_i \sum_{j_{\hat{i}}} \int_0^T ds \int_{\mathbb{R}^d} \mu_s^{D^{N-1}_{\hat{\alpha}_i} \sigma(u)}(d\xi) \big|\mathcal{F}\Lambda(t-s)(\xi)\big|^2 \\
= \int_{[0,T]^{N-1}} d\hat{r}_i \sum_{j_{\hat{i}}} \big\|\Lambda(t-\cdot)\big\|^2_{0,D^{N-1}_{\hat{\alpha}_i}\sigma(u)} \\
\le \liminf_{k\to\infty} \int_{[0,T]^{N-1}} d\hat{r}_i \sum_{\hat{j}_i} \big\|\Lambda_k(t-\cdot)\big\|^2_{0,D^{N-1}_{\hat{\alpha}_i}\sigma(u)} .
\end{aligned}$$

This last term is bounded by a finite constant not depending on k, as can be easily seen using (7.29). Then we conclude by bounded convergence. □

Lemma 7.8

Under the assumptions of Theorem 7.1, for any positive integer $N \ge 1$ and $p \in [1,\infty)$, the sequence $\big(D^N u_n(t,x), n \ge n\big)$ converges in *the topology of* $L^2(\Omega; \mathcal{H}_T^{\otimes N})$ *to the* $\mathcal{H}_T^{\otimes N}$*-valued random vector* $U(t,x)$ *defined by the equation*

$$\begin{aligned} U(t,x) = Z^N(t,x) &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z)\Big[\triangle\big(\sigma, u(s,z)\big) \\ &+ U(s,z)\sigma'\big(u(s,z)\big)\Big] M(ds,dz) \\ &+ \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz)\Big[\triangle\big(b, u(t-s,x-z)\big) \\ &+ U(t-s,x-z)b'\big(u(t-s,x-z)\big)\Big], \end{aligned} \tag{7.30}$$

with $Z^N(t,x)$ *given in Lemma 7.7.*

Proof

We will use an induction argument on N. Let us check that the conclusion is true for $N=1$.

Set $Z^1 := Z$ and

$$\begin{aligned} I_Z^n(t,x) &= Z^{1,n}(t,x) - Z(t,x), \\ I_\sigma^n(t,x) &= \int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s,x-z)\sigma'\big(u_n(s,z)\big) Du_n(s,z) M(ds,dz) \\ &\quad - \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z)\sigma'\big(u(s,z)\big) U(s,z) M(ds,dz), \\ I_b^n(t,x) &= \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz)\Big(b'\big(u_n(t-s,x-z)\big) Du_n(t-s,x-z) \\ &\quad - b'\big(u(t-s,x-z)\big) U(t-s,x-z)\Big). \end{aligned}$$

By the preceding Lemma 7.7,

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\big(\|I_Z^n(t,x)\|_{\mathcal{H}_T}^2\big) = 0.$$

Consider the decomposition

$$E\Big(\big\|I_\sigma^n(t,x)\big\|_{\mathcal{H}_T}^2\Big) \le C\big(D_{1,n}(t,x) + D_{2,n}(t,x) + D_{3,n}(t,x)\big),$$

where

$$\begin{aligned} D_{1,n}(t,x) = E\bigg(\bigg\| \int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s,x-z)\Big[\sigma'\big(u_n(s,z)\big) \\ - \sigma'\big(u(s,z)\big)\Big] Du_n(s,z) M(ds,dz)\bigg\|_{\mathcal{H}_T}^2\bigg), \end{aligned}$$

$$D_{2,n}(t,x) = E\bigg(\Big\|\int_0^t\int_{\mathbb{R}^d} \Lambda_n(t-s,x-z)\sigma'\big(u(s,z)\big)\big[Du_n(s,z) \\ - U(s,z)\big]M(ds,dz)\Big\|^2_{\mathcal{H}_T}\bigg),$$

$$D_{3,n}(t,x) = E\bigg(\Big\|\int_0^t\int_{\mathbb{R}^d} \big[\Lambda_n(t-s,x-z) \\ - \Lambda(t-s,x-z)\big]\sigma'\big(u(s,z)\big)U(s,z)M(ds,dz)\Big\|^2_{\mathcal{H}_T}\bigg).$$

The isometry property of the stochastic integral, Cauchy-Schwarz's inequality and the properties of σ and Λ_n yield

$$D_{1,n}(t,x) \le C \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \Big(E\Big(\big|u_n(s,y)-u(s,y)\big|^4\Big)E\Big(\big\|Du_n(s,y)\big\|^4_{\mathcal{H}_T}\Big)\Big)^{\frac{1}{2}} \\ \times \int_0^t ds\int_{\mathbb{R}^d}\mu(d\xi)\big|\mathcal{F}\Lambda(s)(\xi)\big|^2.$$

Owing to Lemmas 7.5, 7.6 we conclude that

$$\lim_{n\to\infty}\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} D_{1,n}(t,x) = 0.$$

Similarly,

$$D_{2,n}(t,x) \le C\int_0^t ds \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} E\Big(\big\|Du_n(\tau,y)-U(\tau,y)\big\|^2_{\mathcal{H}_T}\Big)J(t-s).$$

Denote by $\bar{U}$ the $\mathcal{H}_T$-valued process $\big\{\sigma'\big(u(s,z)\big)U(s,z),(s,z)\in[0,T]\times\mathbb{R}^d\big\}$. Then, the isometry property yields

$$D_{3,n}(t,x) = \big\|\Lambda_n(t-\cdot,x-*)-\Lambda(t-\cdot,x-*)\big\|^2_{0,\bar{U}} \\ = \int_0^t ds\int_{\mathbb{R}^d}\mu_s^{\bar{U}}(d\xi)\big|\mathcal{F}\psi_n(\xi)-1\big|^2\big|\mathcal{F}\Lambda(t-s)(\xi)\big|^2.$$

Thus, by dominated convergence $\lim_{n\to\infty}\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} D_{3,n}(t,x) = 0$.

For the deterministic integral term, we have

$$E\Big(\big\|I_b^n(t,x)\big\|^2_{\mathcal{H}_T}\Big) \le C\big(b_{1,n}(t,x)+b_{2,n}(t,x)\big),$$

with

$$b_{1,n}(t,x) = E\bigg(\Big\|\int_0^t ds\int_{\mathbb{R}^d}\Lambda(s,dz)\Big[b'\big(u_n(t-s,x-z)\big) \\ - b'\big(u(t-s,x-z)\big)\Big]Du_n(t-s,x-z)\Big\|^2_{\mathcal{H}_T}\bigg),$$

$$b_{2,n}(t,x) = E\left(\left\|\int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz) b'\big(u(t-s,x-z)\big) \right.\right.$$
$$\left.\left. \times \big[Du_n(t-s,x-z) - U(t-s,x-z)\big]\right\|^2_{\mathcal{H}_T}\right).$$

By the properties of the deterministic integral of Hilbert-valued processes, the assumptions on b and Cauchy-Schwarz's inequality we obtain

$$b_{1,n}(t,x) \le \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz) E\left(\Big|b'\big(u_n(t-s,x-z)\big) \right.$$
$$\left. -b'\big(u(t-s,x-z)\big)\Big|^2 \big\|Du_n(t-s,x-z)\big\|^2_{\mathcal{H}_T}\right)$$
$$\le \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \Big(E\big|u_n(s,y)-u(s,y)\big|^4 E\big\|Du_n(s,y)\big\|^4_{\mathcal{H}_T}\Big)^{\frac{1}{2}}$$
$$\times \int_0^t ds \Lambda(s,dz).$$

Thus, $\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} b_{1,n}(t,x) = 0$.

Similar arguments yield

$$b_{2,n}(t,x) \le C\int_0^t ds \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} E\Big(\big\|Du_n(\tau,y) - U(\tau,y)\big\|^2_{\mathcal{H}_T}\Big).$$

Therefore, we have obtained that

$$\sup_{(s,x)\in[0,t]\times\mathbb{R}^d} E\Big(\big\|Du_n(s,x) - U(s,x)\big\|^2_{\mathcal{H}_T}\Big)$$
$$\le C_n + C\int_0^t ds \sup_{(\tau,x)\in[0,s]\times\mathbb{R}^d} E\Big(\big\|Du_n(\tau,x)$$
$$- U(\tau,x)\big\|^2_{\mathcal{H}_T}\Big)\big(J(t-s)+1\big),$$

with $\lim_{n\to\infty} C_n = 0$.

Thus applying Gronwall's Lemma 6.2 we complete the proof.

Assume the induction hypothesis (H_{N-1}) with $p=2$. Then we can proceed in a similar maner than for $N=1$ and complete the proof. We omit the details. □

We are now prepared to give the proof of the main result.

Proof of Theorem 7.1

We follow the same scheme as in the proof of Proposition 7.1. More explicitely, we fix $(t,x)\in[0,T]\times\mathbb{R}^d$ and apply Lemma 7.1 to the sequence $\big(u_n(t,x), n \geq 1\big)$ defined in (7.20). The validity of assumption (a) is ensured by Lemma 7.5. Lemmas 7.6 to 7.8 show that the assumption (b) is also satisfied. Moreover, the process Z^N in Equation (7.3) is given in Lemma 7.7. Hence the proof is complete. □

Comments

The actual presentation of the results of this chapter are not present in previous literature. However, there are several references where particular examples or some pieces of these results are published.

The analysis of the Malliavin differentiability of solutions of SPDE's with coloured noise was first done in reference [40] for the wave equation in spatial dimension $d = 2$. In reference [35] a general setting is presented which covers the stochastic heat equation in any spatial dimension d and the wave equation in dimension $d = 1, 2$. The extension to equations whose fundamental solution is a distribution can be found in references [54] and [55].

7.1 Exercises

7.1.1

Consider the stochastic heat equation in dimension $d \geq 1$ (see Exercise 6.3.5). Prove that if σ, b are $\mathcal{C}^1$ functions with bounded Lipschitz continuous derivatives, then for any $(t,x) \in [0,T] \times \mathbb{R}^d$ the solution $u(t,x)$ belongs to $\mathbb{D}^{1,2}$.

7.1.2

Under the same assumptions of the preceding exercise, prove the same conclusion for the stochastic wave equation in dimension $d = 2$.

7.1.3

Consider the stochastic wave equation in dimension $d = 3$. Prove that if σ, b are $\mathcal{C}^1$ functions with bounded Lipschitz continuous derivatives then, for any $(t,x) \in [0,T] \times \mathbb{R}^d$, the solution $u(t,x)$ belongs to $\mathbb{D}^{1,2}$.

Remark. The purpose of these exercises is to give some insight into Proposition 7.1 in a concrete and simplified setting, and a particular example of the general statement in Theorem 7.1. Actually, a first reading of this chapter could consist in proving results at the level of the first derivative (the first step in the induction assumptions) and then applying them to the examples given here.

Chapter 8

Analysis of the Malliavin Matrix of Solutions of SPDE's

In this Chapter we study the $L^p(\Omega)$-integrability of the inverse of the Malliavin matrix corresponding to the solution of Equation (6.19) at given fixed points $(t, x_1), \dots, (t, x_m)$, $t \in (0, T]$, $x_i \in \mathbb{R}^d$, $i = 1, \dots, m$. The final aim is to combine the results of this and the previous chapter in order to apply Theorem 5.2. That means we shall analyze under which conditions for the coefficients of the equation, the differential operator and the covariance of the noise, the law of the random vector

$$u(t, \underline{x}) = \big(u(t, x_1), \dots, u(t, x_m)\big) \tag{8.1}$$

has an infinitely differentiable density with respect to the Lebesgue measure on $\mathbb{R}^m$.

First we shall assume $m = 1$. In this case the Malliavin matrix is a random variable and the analysis is easier. In a second step we shall give examples where the results can be applied. Finally we shall extend the results to $m > 1$ in some particular cases.

8.1 One dimensional case

For a fixed $(t, x) \in (0, T] \times \mathbb{R}^d$ we consider $u(t, x)$ defined in (6.19). We want to study the validity of the following property:

Property (I) *For any $p > 0$,*

$$E\Big(\big\|Du(t, x)\big\|_{\mathcal{H}_T}^{-p}\Big) < \infty. \tag{8.2}$$

We recall that the Malliavin derivative $Du(t,x)$ satisfies the equation

$$\begin{aligned} Du(t,x) = Z(t,x) + \int_0^t \int_{\mathbb{R}^d} & \Lambda(t-s, x-y) \\ & \times \sigma'\big(u(s,y)\big) Du(s,y) M(ds,dy) \\ + \int_0^t ds & \int_{\mathbb{R}^d} \Lambda(s,dy) \\ & \times b'\big(u(t-s,x-y)\big) Du(t-s,x-y). \end{aligned} \tag{8.3}$$

If $\Lambda(t)$ is absolutely continuous with respect to the Lebesgue measure and we denote by $\Lambda(t,x)$ the density, then $Z(t,x)$ is the $\mathcal{H}_T$-valued random vector $\Lambda(t-\cdot, x-*)\sigma\big(u(\cdot,*)\big)$. For a general Λ satisfying the Hypothesis D, $Z(t,x)$ is obtained as the $L^2(\Omega;\mathcal{H}_T)$-limit of the sequence $\Lambda_n(t-\cdot, x-*)\sigma\big(u(\cdot,*)\big)$, with Λ_n given in (7.18) (see Lemma 7.8, Equation (7.30) with $N=1$).

As in the study of the Malliavin differentiability we shall consider two steps, depending on the regularity properties of the fundamental solution of the equation. The results are given in the next Propositions 8.1 and 8.2.

Proposition 8.1

Suppose that Hypothesis D is satisfied and in addition that the measure $\Lambda(t)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. Moreover, assume that

1) *the coefficients σ and b are $\mathcal{C}^1$ functions with bounded Lipschitz continuous derivatives,*
2) *there exists $\sigma_0 > 0$ such that $\inf\big\{\big|\sigma(z)\big|,\ z \in \mathbb{R}\big\} \geq \sigma_0$,*
3) *there exist θ_i, $C_i > 0$, $i = 1,2,3$, satisfying $\theta_1 < \frac{3}{2}\theta_2 \wedge (\theta_2 + \theta_3)$, such that for any $t \in (0,1)$,*

$$C_1 t^{\theta_1} \leq \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\big|\mathcal{F}\Lambda(s)(\xi)\big|^2 \leq C_2 t^{\theta_2}, \tag{8.4}$$

$$\int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,y)\, dy \leq C_3 t^{\theta_3}. \tag{8.5}$$

Then (I) holds.

Before giving the proof of this Proposition we prove some auxiliary results.

Lemma 8.1

Assume Hypothesis D and that the measure $\Lambda(t)$ is absolutely continuous with respect to the Lebesgue measure. Suppose also that σ satisfies the restriction on the growth

$$\left|\sigma(x)\right| \leq C\big(1+|x|\big).$$

For any $(t,x) \in [0,T]\times\mathbb{R}^d$, define the $\mathcal{H}_t$-valued random variable

$$\tilde{Z}(t,x) = \Lambda(\cdot, x-*)\sigma\big(u(t-\cdot,*)\big).$$

Then, for any $p \in [1,\infty)$

$$E\Big(\big\|\tilde{Z}(t,x)\big\|_{\mathcal{H}_t}^{2p}\Big) \leq C\bigg(\int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\big|\mathcal{F}\Lambda(s)(\xi)\big|^2\bigg)^p. \qquad (8.6)$$

Proof

Hölder's inequality with respect to the non-negative finite measure $\Lambda(s,x-y)\Lambda(s,x-y+z)\,ds\,\Gamma(dz)\,dy$ yields

$$\begin{aligned}
&E\Big(\big\|Z(t,x)\big\|_{\mathcal{H}_t}^{2p}\Big)\\
&= E\bigg(\bigg|\int_0^t ds \int_{\mathbb{R}^d}\Gamma(dz)\int_{\mathbb{R}^d} dy\Lambda(s,x-y)\sigma\big(u(t-s,y)\big)\\
&\qquad \times \Lambda(s,x-y+z)\sigma\big(u(t-s,y-z)\big)\bigg|^p\bigg)\\
&\leq \bigg(\int_0^t ds\int_{\mathbb{R}^d}\Gamma(dz)\int_{\mathbb{R}^d} dy\Lambda(s,x-y)\Lambda(s,x-y+z)\bigg)^{p-1}\\
&\qquad \times \int_0^t ds\int_{\mathbb{R}^d}\Gamma(dz)\int_{\mathbb{R}^d} dy\Lambda(s,x-y)\Lambda(s,x-y+z)\\
&\qquad \times E\Big(\Big|\sigma\big(u(t-s,y)\big)\sigma\big(u(t-s,y-z)\big)\Big|^p\Big)\\
&\leq C\bigg(1+\sup_{(s,z)\in[0,T]\times\mathbb{R}^d} E\Big(\big|u(s,z\big|^{2p}\Big)\bigg)\bigg(\int_0^t ds\int_{\mathbb{R}^d}\Gamma(dz)\big(\Lambda(s)*\tilde{\Lambda}(s)\big)(z)\bigg)^p\\
&\leq C\bigg(\int_0^t ds\int_{\mathbb{R}^d}\mu(d\xi)\big|\mathcal{F}\Lambda(s)(\xi)\big|^2\bigg)^p.
\end{aligned}$$

The proof of (8.6) is complete. □

Lemma 8.2

We assume the same hypothesis on Λ and μ as in the previous lemma. Suppose also that the coefficients σ, b are $\mathcal{C}^1$ functions with bounded Lipschitz continuous derivatives. Then

$$\begin{aligned}\sup_{0\le s\le t}\sup_{x\in\mathbb{R}^d} & E\Big(\big\|D_{t-\cdot,*}u(t-s,x)\big\|_{\mathcal{H}_t}^{2p}\Big)\\ &\le C\left(\int_0^t ds\int_{\mathbb{R}^d}\mu(d\xi)\big|\mathcal{F}\Lambda(s)(\xi)\big|^2\right)^p,\end{aligned} \tag{8.7}$$

for all $t\in[0,T]$ and $p\in[1,\infty)$.

Proof

Owing to the equation (8.3) satisfied by the Malliavin derivative $Du(t,x)$, the proof of (8.7) needs estimates for the $L^{2p}(\Omega;\mathcal{H}_t)$ norm of three terms: the initial condition, the stochastic integral and the path integral. The first one is proved in Lemma 8.1. To obtain the second one we apply (6.8). Finally for the third one we use Jensen's inequality. Then the conclusion follows from Lemma 6.2, taking into account that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\Big(\big\|Du(t,x)\big\|_{\mathcal{H}_T}^{p}\Big)<\infty,$$

for any $p\in[1,\infty)$. □

Lemma 8.3

Property (I) holds if and only if for any $p\in(0,\infty)$ there exists $\epsilon_0>0$, depending on p, such that

$$\int_0^{\epsilon_0}\epsilon^{-(1+p)}P\Big(\big\|Du(t,x)\big\|_{\mathcal{H}_T}^{2}<\epsilon\Big)\,d\epsilon<\infty. \tag{8.8}$$

Proof

It is well known that for any positive random variable,

$$E(F)=\int_0^\infty P(F>\eta)\,d\eta.$$

Apply this formula to $F:=\big\|Du(t,x)\big\|_{\mathcal{H}_T}^{-2p}$. We obtain

$$E\Big(\big\|Du(t,x)\big\|_{\mathcal{H}_T}^{-2p}\Big)=m_1+m_2,$$

with

$$m_1 = \int_0^{\eta_0} P\Big(\big\|Du(t,x)\big\|_{\mathcal{H}_T}^{-2p} > \eta\Big)\, d\eta,$$
$$m_2 = \int_{\eta_0}^{\infty} P\Big(\big\|Du(t,x)\big\|_{\mathcal{H}_T}^{-2p} > \eta\Big)\, d\eta.$$

Clearly, $m_1 \le \eta_0$. The change of variable $\eta = \epsilon^{-p}$ implies

$$\begin{aligned} m_2 &= \int_{\eta_0}^{\infty} P\Big(\big\|Du(t,x)\big\|_{\mathcal{H}_T}^{-2p} > \eta\Big)\, d\eta \\ &= \int_{\eta_0}^{\infty} P\Big(\big\|Du(t,x)\big\|_{\mathcal{H}_T}^{2} < \eta^{-\frac{1}{p}}\Big)\, d\eta \\ &= p \int_0^{\eta_0^{p/2}} \epsilon^{-(1+p)} P\Big(\big\|Du(t,x)\big\|_{\mathcal{H}_T}^{2} < \epsilon\Big)\, d\epsilon. \end{aligned}$$

This finishes the proof. □

Remark 8.1 The process $\big(Du(t,x), (t,x) \in [0,T]\times\mathbb{R}^d\big)$ is $\mathcal{F}_t$-adapted. Hence, by virtue of Lemma 4.2,

$$\big\|Du(t,x)\big\|_{\mathcal{H}_T} = \big\|Du(t,x)\big\|_{\mathcal{H}_t}.$$

Proof of Proposition 8.1

Owing to Lemma 8.3, we have to study the integrability in a neighborhood of zero of the function

$$\Phi(\epsilon) = \epsilon^{-(1+p)} P\Big(\big\|Du(t,x)\big\|_{\mathcal{H}_T}^{2} < \epsilon\Big).$$

Let $\epsilon_1, \delta > 0$ be such that for any $\epsilon \in (0, \epsilon_1]$, $t - \epsilon^\delta > 0$. In view of (8.3) we consider the decomposition

$$\begin{aligned} P\Big(\big\|Du(t,x)\big\|_{\mathcal{H}_T}^{2} < \epsilon\Big) &= P\Big(\big\|Du(t,x)\big\|_{\mathcal{H}_t}^{2} < \epsilon\Big) \\ &\le P^1(\epsilon,\delta) + P^2(\epsilon,\delta), \end{aligned} \tag{8.9}$$

where

$$P^1(\epsilon,\delta) = P\left(\left|\int_{t-\epsilon^\delta}^t dr M(t,r,x)\right| \ge \epsilon\right),$$
$$P^2(\epsilon,\delta) = P\left(\Big\|\Lambda(\cdot, x-*)\sigma\big(u(t-\cdot,*)\big)\Big\|_{\mathcal{H}_{\epsilon^\delta}}^2 < 2\epsilon\right),$$

with $M(t,r,x) = \big\|D_{r,*}u(t,x)\big\|_{\mathcal{H}}^2 - \big\|Z_{r,*}(t,x)\big\|_{\mathcal{H}}^2$.

Indeed,

$$\begin{aligned}\left\|Du(t,x)\right\|_{\mathcal{H}_t}^2 &\geq \int_{t-\epsilon^\delta}^t dr \left\|D_{r,*}u(t,x)\right\|_{\mathcal{H}}^2\\ &= \int_{t-\epsilon^\delta}^t dr \left\|Z_{r,*}(t,x)\right\|_{\mathcal{H}}^2 + \int_{t-\epsilon^\delta}^t dr M(t,r,x)\end{aligned}$$

and (8.9) follows from the triangular inequality.

Let us first consider the term $P^1(\epsilon,\delta)$. By *Chebychev's inequality*, for every $q \geq 1$ we have that

$$P^1(\epsilon,\delta) \leq \epsilon^{-q} E\left(\left|\int_{t-\epsilon^\delta}^t dr M(t,r,x)\right|^q\right) \leq C\epsilon^{-q}\sum_{k=1}^5 T_k, \tag{8.10}$$

with

$$\begin{aligned}T_1 = E\Bigg(\Bigg|\int_{t-\epsilon^\delta}^t dr \Big\langle Z_{r,*}(t,x), \int_{t-\epsilon^\delta}^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z) D_{r,*}u(s,z)\\ \times \sigma'\big(u(s,z)\big) M(ds,dz)\Big\rangle_{\mathcal{H}}\Bigg|^q\Bigg),\end{aligned}$$

$$\begin{aligned}T_2 = E\Bigg(\Bigg|\int_{t-\epsilon^\delta}^t dr \Big\langle Z_{r,*}(t,x), \int_{t-\epsilon^\delta}^t ds \int_{\mathbb{R}^d} \Lambda(t-s,dz) D_{r,*}u(s,x-z)\\ \times b'\big(u(s,x-z)\big)\Big\rangle_{\mathcal{H}}\Bigg|^q\Bigg),\end{aligned}$$

$$\begin{aligned}T_3 = E\Bigg(\Bigg|\int_{t-\epsilon^\delta}^t dr \Bigg\|\int_{t-\epsilon^\delta}^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z) D_{r,*}u(s,z)\\ \times \sigma'\big(u(s,z)\big) M(ds,dz)\Bigg\|_{\mathcal{H}}^2\Bigg|^q\Bigg),\end{aligned}$$

$$\begin{aligned}T_4 = E\Bigg(\Bigg|\int_{t-\epsilon^\delta}^t dr\\ \times \Big\langle \int_{t-\epsilon^\delta}^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z) D_{r,*}u(s,z)\sigma'\big(u(s,z)\big) M(ds,dz),\\ \int_{t-\epsilon^\delta}^t ds \int_{\mathbb{R}^3} \Lambda(t-s,dz) D_{r,*}u(s,x-z) b'\big(u(s,x-z)\big)\Big\rangle_{\mathcal{H}}\Bigg|^q\Bigg),\end{aligned}$$

$$\begin{aligned}T_5 = E\Bigg(\Bigg|\int_{t-\epsilon^\delta}^t dr \Bigg\|\int_{t-\epsilon^\delta}^t ds \int_{\mathbb{R}^d} \Lambda(t-s,dz) D_{r,*}u(s,x-z)\\ \times b'\big(u(s,x-z)\big)\Bigg\|_{\mathcal{H}}^2\Bigg|^q\Bigg).\end{aligned}$$

The Cauchy-Schwarz inequality yields

$$T_1 \le T_{11}^{1/2} T_{12}^{1/2},$$

with

$$T_{11} = E\left(\left|\int_{t-\epsilon^\delta}^t dr \|Z_{r,*}(t,x)\|_{\mathcal{H}}^2\right|^q\right),$$

$$T_{12} = E\left(\left|\int_{t-\epsilon^\delta}^t dr \left\|\int_{t-\epsilon^\delta}^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z) \times D_{r,*}u(s,z)\sigma'\big(u(s,z)\big)M(ds,dz)\right\|_{\mathcal{H}}^2\right|^q\right).$$

By Lemma 8.1 and (8.4),

$$T_{11} \le C\left(\int_0^{\epsilon^\delta} ds \int_{\mathbb{R}^d} \mu(d\xi)\big|\mathcal{F}\Lambda(s)(\xi)\big|^2\right)^q \le C\epsilon^{q\delta\theta_2}. \tag{8.11}$$

Clearly,

$$T_{12} = E\left(\left\|\int_{t-\epsilon^\delta}^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z) \times D_{t-\cdot,*}u(s,z)\sigma'\big(u(s,z)\big)M(ds,dz)\right\|_{\mathcal{H}_{\epsilon^\delta}}^{2q}\right).$$

Here we apply Theorem 6.1 to

$$\mathcal{K} := \mathcal{H}_{\epsilon^\delta}, \qquad K(s,z) := D_{t-\cdot,*}u(s,z)\sigma'\big(u(s,z)\big) \quad \text{and} \quad S := \Lambda.$$

Thus, Lemma 8.2 and (8.4) ensure

$$T_{12} \le C\left(\int_0^{\epsilon^\delta} ds \int_{\mathbb{R}^d} \mu(d\xi)\big|\mathcal{F}\Lambda(s)(\xi)\big|^2\right)^{2q} \le \epsilon^{2q\delta\theta_2}. \tag{8.12}$$

Hence,

$$T_1 \le C\epsilon^{\frac{3}{2}q\delta\theta_2}. \tag{8.13}$$

We now consider the term

$$T_{22} := E\left(\left\|\int_{t-\epsilon^\delta}^t ds \int_{\mathbb{R}^d} \Lambda(t-s, dz) D_{t-\cdot,*}u(s, x-z) b'\big(u(s,x-z)\big)\right\|_{\mathcal{H}_{\epsilon^\delta}}^{2q}\right).$$

Jensen's inequality and then Hölder's inequality with respect to the finite measure $\Lambda(t-s,dz)\,ds$ on $[t-\epsilon^\delta,t]\times\mathbb{R}^3$ yield

$$T_{22} \le \left(\int_{t-\epsilon^\delta}^t ds \int_{\mathbb{R}^d} \Lambda(t-s,dz)\right)^{2q-1} \times E\left(\int_{t-\epsilon^\delta}^t ds \int_{\mathbb{R}^d} \Lambda(t-s,dz)\Big\| D_{t-\cdot,*}u(s,x-z)b'\big(u(s,x-z)\big)\Big\|_{\mathcal{H}_{\epsilon^\delta}}^{2q}\right).$$

From (8.5) we obtain

$$\int_{t-\epsilon^\delta}^t ds \int_{\mathbb{R}^d} \Lambda(t-s,dz) = \int_0^{\epsilon^\delta} ds \int_{\mathbb{R}^d} dz \Lambda(s,z) \le C\epsilon^{\theta_3\delta},$$

Then, since b' is bounded, Lemma 8.2 and (8.4) imply

$$T_{22} \le C\epsilon^{q\delta(2\theta_3+\theta_2)}. \tag{8.14}$$

The Cauchy-Schwarz inequality and the estimates (8.11), (8.12), (8.14) yield

$$\begin{aligned} T_2 &\le T_{11}^{1/2}T_{22}^{1/2} \le C\epsilon^{q\delta(\theta_2+\theta_3)},\\ T_3 &= T_{12} \le C\epsilon^{2q\delta\theta_2},\\ T_4 &\le T_{12}^{1/2}T_{22}^{1/2} \le C\epsilon^{q\delta(\frac{3}{2}\theta_2+\theta_3)},\\ T_5 &= T_{22} \le C\epsilon^{q\delta(2\theta_3+\theta_2)}. \end{aligned} \tag{8.15}$$

Therefore, (8.10)-(8.15) imply

$$P^1(\epsilon,\delta) \le C\epsilon^{q\left(-1+(\frac{3}{2}\delta\theta_2)\wedge(\delta(\theta_2+\theta_3))\right)}.$$

Consequently, for any $\epsilon_0>0$ condition $\int_0^{\epsilon_0} P^1(\epsilon,\delta)\epsilon^{-(1+p)}d\epsilon<\infty$ holds if

$$\frac{1}{\delta} < \frac{q\left(\frac{3}{2}\,\theta_2\wedge(\theta_2+\theta_3)\right)}{p+q}. \tag{8.16}$$

We now study the term $P^2(\epsilon,\delta)$. Our purpose is to chose some positive δ such that for ϵ sufficiently small the set $\big(\|\Lambda(\cdot,x-*)\sigma(u(t-\cdot,*))\|_{\mathcal{H}_{\epsilon^\delta}}^2 < 2\epsilon\big)$ is empty and therefore $P^2(\epsilon,\delta)=0$.

The assumption 2) yields

$$\Big\|\Lambda(r,x-*)\sigma\big(u(t-r,*)\big)\Big\|_{\mathcal{H}}^2 \ge \sigma_0^2\int_{\mathbb{R}^d}\mu(d\xi)\big|\mathcal{F}\Lambda(r)(\xi)\big|^2.$$

Hence, the lower bound in (8.4) implies

$$\Big\|\Lambda(\cdot,x-*)\sigma\big(u(t-\cdot,*)\big)\Big\|_{\mathcal{H}_{\epsilon^\delta}}^2 \ge C_1\sigma_0^2\epsilon^{\delta\theta_1}.$$

Let $\delta > 0$ be such that

$$\delta\theta_1 < 1. \tag{8.17}$$

Set $\epsilon_2 := \left(\frac{C_1}{2}\,\sigma_0^2\right)^{1/(1-\delta\theta_1)}$. Then for any $\epsilon \le \epsilon_2$ we have that $P^2(\epsilon,\delta) = 0$.

Summarizing the restrictions imposed so far we obtain (see (8.16) and (8.17))

$$\theta_1 < \frac{1}{\delta} < \frac{\left(\frac{3}{2}\,\theta_2 \wedge (\theta_2+\theta_3)\right)q}{p+q}. \tag{8.18}$$

Fix $q_0 \in (1,\infty)$ such that

$$\theta_1 < \frac{\left(\frac{3}{2}\,\theta_2 \wedge (\theta_2+\theta_3)\right)q_0}{p+q_0}.$$

Since by assumption $\theta_1 < \frac{3}{2}\theta_2 \wedge (\theta_2+\theta_3)$ this is always possible. Then let δ_0 be such that (8.18) holds with $q := q_0$. Let $\epsilon_1 > 0$ be such that for any $\epsilon \in (0,\epsilon_1]$, $t - \epsilon^{\delta_0} > 0$.

The preceding arguments with $\delta := \delta_0$ and $q := q_0$ prove that the function $\Psi(\epsilon) = \epsilon^{-(1+p)} P\left(\|Du(t,x)\|^2_{\mathcal{H}_T} < \epsilon\right)$ is integrable on $(0,\epsilon_0)$, with $\epsilon_0 = \epsilon_1 \wedge \epsilon_2$. This finishes the proof of the Proposition. □

Let us now consider the case where Λ satisfies Hypothesis D, without further smoothness properties.

Proposition 8.2

Suppose that Hypothesis D is satisfied and also that:

1) *the coefficients σ and b are $\mathcal{C}^1$ functions with bounded Lipschitz continuous derivatives,*
2) *there exists $\sigma_0 > 0$ such that* $\inf\left\{|\sigma(z)|, z \in \mathbb{R}\right\} \ge \sigma_0$,
3) $\theta_4 < \theta_1 < \frac{3}{2}\theta_2 \wedge (\theta_2+\theta_3)$, *such that for any* $t \in (0,1)$,

$$C_1 t^{\theta_1} \le \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\left|\mathcal{F}\Lambda(s)(\xi)\right|^2 \le C_2 t^{\theta_2}, \tag{8.19}$$

$$\int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dy) \le C_3 t^{\theta_3}, \tag{8.20}$$

$$\int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)|\xi| \left|\mathcal{F}\Lambda(s)(\xi)\right|^2 \le C_4 t^{\theta_4}, \tag{8.21}$$

$$\int_0^t ds \int_{\mathbb{R}^d} \mu_s^{\bar{\sigma}}(d\xi)|\xi| \left|\mathcal{F}\Lambda(s)(\xi)\right|^2 \le C_4 t^{\theta_5}. \tag{8.22}$$

where $\bar{\sigma}(s,x) = \sigma\big(u(t-s,x)\big)$, $0 \le s \le t$.

Then (I) holds.

We recall that $\mu_s^{\bar{\sigma}}$ is the spectral measure of the finite measure

$$\Gamma_s^{\bar{\sigma}}(dz) = E\Big(\bar{\sigma}\big(u(t-s,0)\big)\bar{\sigma}\big(u(t-s,z)\big)\Big)\Gamma(dz),$$

(see Section 7.1).

The difference between the proof of this proposition and the preceding one lies on the analysis of the term $P^2(\epsilon,\delta)$, where we use a mollifying procedure. More precisely, as in Chapter 7, let ψ be a non-negative function in $\mathcal{C}^\infty(\mathbb{R}^d)$ with support contained in the unit ball of $\mathbb{R}^d$ and such that $\int_{\mathbb{R}^d} \psi(x)\,dx = 1$. For any $\eta > 0$, set $\psi_\eta(x) = \eta^d \psi(\eta x)$ and $\Lambda_\eta = \psi_\eta * \Lambda$. We shall need a technical result.

Lemma 8.4

We have the following upper bound:

$$\left|\mathcal{F}(\Lambda_\eta - \Lambda)(t)(\xi)\right|^2 \le 4\pi \left|\mathcal{F}\Lambda(t)(\xi)\right|^2 |\xi|\eta^{-1}, \tag{8.23}$$

for any $t \in [0,T]$ *and* $\xi \in \mathbb{R}^d$.

Proof

The definition of the Fourier transform and a change of variables yield

$$\mathcal{F}\psi_\eta(\xi) = \int_{\mathbb{R}^d} \psi_\eta(x) \exp(-2i\pi x\cdot\xi)\,dx$$
$$= \int_{\mathbb{R}^d} \psi(y) \exp\left(-2i\pi \frac{y}{\eta}\cdot\xi\right) dy,$$

where the notation "$\cdot$" means the inner product in $\mathbb{R}^d$. Consequently,

$$\begin{aligned}\left|\mathcal{F}\psi_\eta(\xi)-1\right|^2 &= \left|\int_{\mathbb{R}^d}\psi(y)\left(\exp\left(-2i\pi\frac{y}{\eta}\cdot\xi\right)-1\right)dy\right|^2\\ &= \left|\int_{|y|\le 1}\psi(y)\left(\exp\left(-2i\pi\frac{y}{\eta}\cdot\xi\right)-1\right)dy\right|^2\\ &\le \sup_{|y|\le 1}\left|\exp\left(-2i\pi\frac{y}{\eta}\cdot\xi\right)-1\right|^2\\ &= 2\sup_{|y|\le 1}\Big(1-\cos\big(2\pi(y\cdot\xi)\eta^{-1}\big)\Big).\end{aligned}$$

A first-order Taylor expansion of the function $\cos(x)$ in a neighborhood of zero yields for the last term the upper bound

$$2\sup_{|y|\le 1}\Big(2\pi(y\cdot\xi)\eta^{-1}\sin\big(2\pi(y\cdot\xi)\eta_0\big)\Big)$$

with $\eta_0\in(0,\eta^{-1})$. Therefore,

$$\left|\mathcal{F}\psi_\eta(\xi)-1\right|^2\le 4\pi|\xi|\eta^{-1}.$$

This proves (8.23). □

Proof of Proposition 8.2

As in the proof of Proposition 8.1 we shall use Lemma 8.3 and consider the decomposition

$$P\Big(\|Du(t,x)\|^2_{\mathcal{H}_T}<\epsilon\Big)\le P^1(\epsilon,\delta)+P^2(\epsilon,\delta),$$

where

$$\begin{aligned}P^1(\epsilon,\delta)&=P\left(\left|\int_{t-\epsilon^\delta}^t dr M(t,r,x)\right|\ge\epsilon\right),\\ P^2(\epsilon,\delta)&=P\Big(\|\bar Z(t,x)\|^2_{\mathcal{H}_{\epsilon^\delta}}<2\epsilon\Big),\end{aligned}$$

$\bar Z(t,x)$ is the $L^2(\Omega;\mathcal{H}_T)$-limit of the sequence $\Lambda_n(\cdot,x-*)\sigma\big(u(t-\cdot,*)\big)$. Notice that $\bar Z_{\cdot,*}(t,x)=Z_{t-\cdot,*}(t,x)$.

We obtain that $\int_0^{\epsilon_0}P^1(\epsilon,\delta)\epsilon^{-(p+1)}\,d\epsilon<\infty$ if the restriction (8.16) is satisfied.

The analysis of $P^2(\epsilon,\delta)$ cannot be carried out as in Proposition 8.1. In fact the process $Z(t,x)$ is no more a product of a deterministic function and a process. We overcome this problem by smoothing the fundamental solution Λ and controlling the error made in this approximation. To this end, we introduce a further decomposition, as follows,

$$P^2(\epsilon,\delta) \le P^{2,1}(\epsilon,\delta,\nu) + P^{2,2}(\epsilon,\delta,\nu),$$

where $\nu > 0$ and

$$P^{2,1}(\epsilon,\delta,\nu) = P\Big(\Big\|\Lambda_{\epsilon^{-\nu}}(\cdot,x-*)\sigma\big(u(t-\cdot,*)\big)\Big\|^2_{\mathcal{H}_{\epsilon^\delta}} < 6\epsilon\Big),$$
$$P^{2,2}(\epsilon,\delta,\nu) = P\Big(\Big\|Z_{t-\cdot,*}(t,x) - \Lambda_{\epsilon^{-\nu}}(\cdot,x-*)\sigma\big(u(t-\cdot,*)\big)\Big\|^2_{\mathcal{H}_{\epsilon^\delta}} \ge \epsilon\Big).$$

Let us start with the study of the term $P^{2,1}(\epsilon,\delta,\nu)$. Our purpose is to choose some positive δ and ν such that for ϵ sufficiently small the set

$$\Big(\Big\|\Lambda_{\epsilon^{-\nu}}(\cdot,x-*)\sigma\big(u(t-\cdot,*)\big)\Big\|^2_{\mathcal{H}_{\epsilon^\delta}} < 6\epsilon\Big)$$

is empty and therefore $P^{2,1}(\epsilon,\delta,\nu) = 0$.

The assumption 2) yields

$$\begin{aligned}\Big\|\Lambda_{\epsilon^{-\nu}}(r,x-*)\sigma\big(u(t-r,*)\big)\Big\|^2_{\mathcal{H}} &\ge \sigma_0^2 \int_{\mathbb{R}^d} \mu(d\xi)\big|\mathcal{F}\Lambda_{\epsilon^{-\nu}}(r)(\xi)\big|^2 \\ &\ge \sigma_0^2\left(\frac12\int_{\mathbb{R}^d}\mu(d\xi)\big|\mathcal{F}\Lambda(r)(\xi)\big|^2 \right.\\ &\qquad \left. - \int_{\mathbb{R}^d}\mu(d\xi)\big|\mathcal{F}(\Lambda_{\epsilon^{-\nu}}-\Lambda)(r)(\xi)\big|^2\right).\end{aligned}$$

Lemma 8.4 and the bounds (8.19), (8.21) yield

$$\begin{aligned}\Big\|\Lambda_{\epsilon^{-\nu}}(\cdot,x-*)\sigma\big(u(t-\cdot,*)\big)\Big\|^2_{\mathcal{H}_{\epsilon^\delta}} &\ge \sigma_0^2\left(\frac12\int_0^{\epsilon^\delta} dr\int_{\mathbb{R}^d}\mu(d\xi)\big|\mathcal{F}\Lambda_3(r)(\xi)\big|^2\right.\\ &\qquad \left. -4\pi\epsilon^\nu\int_0^{\epsilon^\delta} ds\int_{\mathbb{R}^d}\mu(d\xi)|\xi|\,\big|\mathcal{F}\Lambda(r)(\xi)\big|^2\right)\\ &\ge \sigma_0^2\left(\frac12 C_1\epsilon^{\theta_1\delta} - C_2\epsilon^{\nu+\delta\theta_4}\right),\end{aligned}$$

for some positive constants C_1, C_2.

Let $\nu, \delta > 0$ be such that

$$\frac{\theta_1 - \theta_4}{\nu} < \frac{1}{\delta}\,, \tag{8.24}$$

then $\frac{1}{2}C_1\epsilon^{\theta_1\delta} - C_2\epsilon^{\nu+\delta\theta_4} \geq \frac{C_1}{4}\epsilon^{\theta_1\delta}$, for all $\epsilon \leq \epsilon_2 := \left(\frac{C_1}{4C_2}\right)^{1/(\nu-\delta(\theta_1-\theta_4))}$. Thus, for any $\epsilon \leq \epsilon_2$,

$$\left\|\Lambda_{\epsilon^{-\nu}}(\cdot, x-*)\sigma\big(u(t-\cdot,*)\big)\right\|^2_{\mathcal{H}_{\epsilon^\delta}} \geq \sigma_0^2\frac{1}{4}\,C_1\epsilon^{\theta_1\delta}.$$

Moreover, the condition

$$\theta_1\delta < 1, \tag{8.25}$$

implies $6\epsilon < \sigma_0^2\frac{C_1}{4}\epsilon^{\theta_1\delta}$, for $\epsilon \leq \epsilon_3 := \left(\frac{C_1\sigma_0^2}{24}\right)^{1/(1-\theta_1\delta)}$.

Hence, if $\nu, \delta > 0$ satisfy (8.24) and (8.25) then $P^{2,1}(\epsilon,\delta,\nu) = 0$, for any $\epsilon \leq \epsilon_2 \wedge \epsilon_3$.

Consider now the term $P^{2,2}(\epsilon,\delta,\nu)$. By Chebychev's inequality, Lemma 8.4 and (8.22) we have that

$$\begin{aligned}
P^{2,2}(\epsilon,\delta,\nu) &\leq \epsilon^{-1}E\left(\left\|Z_{t-\cdot,*}(t,x) - \Lambda_{\epsilon^{-\nu}}(\cdot, x-*)\sigma\big(u(t-\cdot,*)\big)\right\|^2_{\mathcal{H}_{\epsilon^\delta}}\right)\\
&= \epsilon^{-1}\int_0^{\epsilon^\delta} ds \int_{\mathbb{R}^d} \mu_s^{\bar\sigma}(d\xi)\left|\mathcal{F}\big(\Lambda(s) - \Lambda_{\epsilon^{-\nu}}(s)\big)(\xi)\right|^2\\
&\leq 4\pi\epsilon^{-1+\nu}\int_0^{\epsilon^\delta} ds \int_{\mathbb{R}^d} \mu_s^{\bar\sigma}(d\xi)\left|\mathcal{F}\Lambda(s)(\xi)\right|^2 \leq C\epsilon^{-1+\nu+\delta\theta_5},
\end{aligned}$$

for some positive constant C.

Thus, $\int_0^{\epsilon_0} \epsilon^{-(1+p)}P^{2,2}(\epsilon,\delta,\nu)\,d\epsilon < \infty$ if

$$-1 - p + \nu + \delta\theta_5 > 0. \tag{8.26}$$

We finish the proof by analyzing the intersection of conditions (8.16), (8.24)-(8.26). We recall that $p \in [0,\infty)$ is fixed.

Choose $\nu > 0$ such that

$$\nu > \frac{\theta_1 - \theta_4}{\theta_1}\,. \tag{8.27}$$

We are assuming that $\theta_1 > \theta_4 > 0$, therefore such a choice is possible. Then conditions (8.16), (8.25), (8.26) are equivalent to

$$\theta_1 < \frac{1}{\delta} < \frac{q\left(\frac{3}{2}\theta_2 \wedge (\theta_2 + \theta_3)\right)}{p+q}\,, \tag{8.28}$$

$$-1 - p + \nu + \delta\theta_5 > 0. \tag{8.29}$$

Let us now choose $q_0 \in (1, \infty)$ such that

$$\theta_1 < \frac{q_0\big(\frac{3}{2}\theta_2 \wedge (\theta_2 + \theta_3)\big)}{p + q_0}.$$

The condition $\theta_1 < \frac{3}{2}\theta_2 \wedge (\theta_2 + \theta_3)$ allows this choice. Then let $\delta > 0$ satisfying (8.28) with $q = q_0$. For this δ choose $\nu > 0$ sufficiently large such that (8.27) and (8.29) hold true.

The proof of the proposition is complete. □

If our purpose were limited to studying the existence of a density for the probability law of $u(t,x)$, it would suffice to study the validity of the property $\big\|Du(t,x)\big\|_{\mathcal{H}_T} > 0$, a.s. A sufficient condition for this is

Property (I') *There exists $p > 0$ such that $E\big(\|Du(t,x)\|_{\mathcal{H}_T}^{-p}\big) < \infty$.*

In fact, (I') implies that the positive random variable $\big\|Du(t,x)\big\|_{\mathcal{H}_T}^{-p}$ is finite a.s. and hence, $\big\|Du(t,x)\big\|_{\mathcal{H}_T} > 0$, a.s.

We leave as an exercise for the reader to give a direct proof of the next proposition, by adapting some arguments of Propositions 8.1 and 8.2.

Proposition 8.3

Suppose that Hypothesis D is satisfied and also that:

1) *the coefficients σ and b are $\mathcal{C}^1$ functions with bounded Lipschitz continuous derivatives,*
2) *there exists $\sigma_0 > 0$ such that $\inf\big\{|\sigma(z)|, z \in \mathbb{R}\big\} \geq \sigma_0$,*
3) *there exist $\theta_i, C_i > 0$, $i = 1,2,3,4$, satisfying $\theta_4 < \theta_1 < \frac{3}{2}\theta_2 \wedge (\theta_2 + \theta_3)$ such that for any $t \in (0,1)$,*

$$C_1 t^{\theta_1} \leq \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\big|\mathcal{F}\Lambda(s)(\xi)\big|^2 \leq C_2 t^{\theta_2},$$

$$\int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dy) \leq C_3 t^{\theta_3},$$

$$\int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)|\xi|\,\big|\mathcal{F}\Lambda(s)(\xi)\big|^2 \leq C_4 t^{\theta_4},$$

$$\int_0^t ds \int_{\mathbb{R}^d} \mu_s^{\bar{\sigma}}(d\xi)|\xi|\,\big|\mathcal{F}\Lambda(s)(\xi)\big|^2 \leq C_4 t^{\theta_5}.$$

where $\bar{\sigma}(s,x) = \sigma\big(u(t-s,x)\big)$, $0 \leq s \leq t$.

Then (I') holds.

8.2 Examples

We study in this section two important examples of stochastic partial differential equations: the wave and heat equations. We shall check that the assumptions of Proposition 8.1 are satisfied by the heat equation in any spatial dimension $d \geq 1$ and by the wave equation in dimension $d = 1, 2$, while Proposition 8.2 applies to the wave equation for $d = 3$.

The next assumption shall play a relevant role.

Condition ($\mathbf{H}_\eta$) *There exists $\eta \in (0, 1]$ such that*

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{\left(1 + |\xi|^2\right)^\eta} < \infty. \tag{8.30}$$

Notice that for $\eta = 1$, this is (6.16).

We begin with the analysis of the heat operator.

Lemma 8.5

Let Λ be the fundamental solution of $L_1 = 0$, with $L_1 = \partial_t - \triangle_d$.

1) Assume that condition (6.16) holds; then for any $t \geq 0$ there exists a positive constant $C > 0$, not depending on t, such that

$$Ct \leq \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) \left|\mathcal{F}\Lambda(s)(\xi)\right|^2. \tag{8.31}$$

2) Suppose that (8.30) holds for some $\eta \in (0, 1)$. Then for any $t \geq 0$ there exists a constant $C > 0$, not depending on t, and $\beta \in (0, 1 - \eta]$ such that

$$\int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) \left|\mathcal{F}\Lambda(s)(\xi)\right|^2 \leq Ct^\beta. \tag{8.32}$$

3) For any $t \geq 0$, there exists a positive constant C, not depending on t, such that

$$\int_0^t ds \int_{\mathbb{R}^d} dy \Lambda(s, y) \leq Ct. \tag{8.33}$$

PROOF

We recall that $\mathcal{F}\Lambda(t)(\xi) = \exp\left(-2\pi^2 t|\xi|^2\right)$. The proof of 1) follows immediately from the lower bound of (6.15). For the proof of 2) we consider the decomposition

$$\int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\left|\mathcal{F}\Lambda(s)(\xi)\right|^2 \leq T_1(t) + T_2(t),$$

with

$$T_1(t) = \int_0^t ds \int_{|\xi|\leq 1} \mu(d\xi)\left|\mathcal{F}\Lambda(s)(\xi)\right|^2,$$
$$T_2(t) = \int_0^t ds \int_{|\xi|>1} \mu(d\xi)\left|\mathcal{F}\Lambda(s)(\xi)\right|^2.$$

Since μ is a tempered measure, it is finite on any compact set. Thus

$$T_1(t) \leq \mu\left(|\xi| \leq 1\right)t.$$

To study $T_2(t)$ we apply the inequality $1 - \exp(-x) \leq x$, valid for any $x \geq 0$. We obtain,

$$\begin{aligned} T_2(t) &= \int_{|\xi|>1} \mu(d\xi) \frac{\left(1-\exp(-4\pi^2 t|\xi|^2)\right)^{\eta}\left(1-\exp\left(-4\pi^2 t|\xi|^2\right)\right)^{1-\eta}}{4\pi^2|\xi|^2} \\ &\leq 2^{\eta} \int_{|\xi|>1} \mu(d\xi) \frac{\left(1-\exp\left(-4\pi^2 t|\xi|^2\right)\right)^{1-\eta}}{4\pi^2|\xi|^2} \\ &\leq 2^{-\eta}\pi^{-2\eta} \int_{|\xi|>1} \mu(d\xi) \frac{t^{1-\eta}}{|\xi|^{2\eta}} \\ &\leq \pi^{-2\eta} t^{1-\eta} \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{\left(1+|\xi|^2\right)^{\eta}}. \end{aligned}$$

The statement 3) is trivial. Indeed by its very definition $\int_{\mathbb{R}^d} \Lambda(s,y)\,dy \leq C$. □

As a consequence of the previous Lemma we can now state the result concerning the heat equation.

Theorem 8.1

Let $u(t,x)$ be the solution of Equation (6.13) with $L := \partial_t - \triangle_d$ at a fixed point $(t,x) \in (0,T] \times \mathbb{R}^d$. Suppose that

1) the coefficients σ, b are $\mathcal{C}^\infty$ functions with bounded derivatives of any order greater or equal than one, *2) there exists $\sigma_0 > 0$ such that $\inf\{|\sigma(z)|, z \in \mathbb{R}\} \geq \sigma_0$,*

3) there exists $\eta \in (0, \frac{1}{3})$ such that condition (H_η) holds.

Then, the law of $u(t,x)$ has an infinite differentiable density with respect to the Lebesgue measure on $\mathbb{R}$.

Proof

It is based on the criterium given in Theorem 5.2. The validity of assumption a) of this Theorem is ensured by Proposition 7.1. Let us now check assumption b). By virtue of Lemma 8.5 the hypotheses 3) of Proposition 8.1 hold with $\theta_1 = 1$, $\theta_2 = 1 - \eta$ and $\theta_3 = 1$.

These parameters satisfy the restriction $\theta_1 < \frac{3}{2}\theta_2 \wedge (\theta_2 + \theta_3)$ if $\eta \in (0, \frac{1}{3})$. □

The next step is to study the stochastic wave equation with spatial parameter $d = 1, 2$. We begin with some auxiliary results whose validity extends to any dimension $d \geq 1$.

Lemma 8.6

Let Λ be the fundamental solution of $L_2 = 0$ with $L_2 = \partial^2_{tt} - \triangle_d$, $d \geq 1$.

1) Assume that (6.16) holds; then for any $t \geq 0$ we have

$$C_1(t \wedge t^3) \leq \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) \left|\mathcal{F}\Lambda(s)(\xi)\right|^2 \leq C_2(t + t^3), \tag{8.34}$$

where C_i, $i = 1, 2$ are positive constants independent of t. In particular, for $t \in [0, 1)$,

$$C_1 t^3 \leq \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) \left|\mathcal{F}\Lambda(s)(\xi)\right|^2 \leq C_2 t. \tag{8.35}$$

2) Suppose that (H_η) holds for some $\eta \in (0, 1)$. Then for any $t \in [0, T]$,

$$\int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) \left|\mathcal{F}\Lambda(s)(\xi)\right|^2 \leq C t^{3-2\eta}, \tag{8.36}$$

where C is a positive constant depending on η and T.

3) Let $d \in \{1,2,3\}$. Then there exists a positive constant independent of t such that

$$\int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dy) \le Ct^2. \tag{8.37}$$

Proof

The estimates (8.34) follow from (6.17) and (8.35) is a trivial consequence of (8.34).

Let us check (8.36). Set

$$\int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) \left|\mathcal{F}\Lambda(s)(\xi)\right|^2 = R_1(t) + R_2(t),$$

with

$$R_1(t) = \int_0^t ds \int_{|\xi|\le 1} \mu(d\xi) \frac{\sin^2\left(2\pi s|\xi|\right)}{\left(2\pi|\xi|\right)^2},$$

$$R_2(t) = \int_0^t ds \int_{|\xi|> 1} \mu(d\xi) \frac{\sin^2\left(2\pi s|\xi|\right)}{\left(2\pi|\xi|\right)^2}.$$

Since $\sin x \le x$, we clearly have

$$R_1(t) \le \mu\{|\xi|\le 1\}\frac{t^3}{3}.$$

For $R_2(t)$ we have

$$\begin{aligned} R_2(t) &\le \int_0^t ds \int_{|\xi|>1} \mu(d\xi) \frac{\left(\sin(2\pi s|\xi|\right)^{2(1-\eta)}}{\left(2\pi|\xi|\right)^2} \\ &\le \int_0^t ds \int_{|\xi|>1} \frac{\mu(d\xi)}{4\pi^2|\xi|^2} \left(2\pi s|\xi|\right)^{2(1-\eta)} \\ &\le \frac{1}{\pi^{2\eta}(3-2\eta)} \left(\int_{|\xi|>1} \frac{\mu(d\xi)}{\left(1+|\xi|^2\right)^\eta}\right) t^{3-2\eta}. \end{aligned}$$

Therefore, we obtain the upper bound (8.36) with

$$C = \frac{\mu\{|\xi|\le 1\}}{3} T^{2\eta} + \frac{1}{2\pi^{2\eta}(3-2\eta)} \int_{|\xi|>1} \frac{\mu(d\xi)}{\left(1+|\xi|^2\right)^\eta}.$$

To prove 3) we have to look at the diferent values of d. For $d = 1$, $\Lambda(s)$ is the measure defined by $\Lambda(s, dy) = \left(\frac{1}{2}\mathbf{1}_{|y|<s}\right) dy$. Thus,

$$\int_0^t ds \int_{\mathbb{R}} \Lambda(s, dy) = \frac{t^2}{2}.$$

For $d = 2$,

$$\Lambda(s, dy) = \left(\frac{1}{2\pi}\left(s^2 - |y|^2\right)^{-\frac{1}{2}} \mathbf{1}_{|y|\le s}\right) dy.$$

Thus, a direct computation yields

$$\int_0^t ds \int_{\mathbb{R}^2} \Lambda(s, dy) = \frac{t^2}{2}.$$

Finally, for $d = 3$, $\Lambda(s) = 1/(4\pi s)\sigma_s$, where σ_s denotes the uniform measure on the 3-dimensional sphere of radius s. Therefore,

$$\int_0^t ds \int_{\mathbb{R}^3} \Lambda(s, dy) = \int_0^t s\, ds = \frac{t^2}{2}. \qquad \square$$

The above lemma allows to study the existence and smoothness of density for the stochastic wave equation with $d = 1, 2$, as follows.

Theorem 8.2

Let $u(t, x)$ be the solution of Equation (6.13) with $L := \partial_{tt}^2 - \triangle_d$, $d = 1, 2$, at a fixed point $(t, x) \in (0, T] \times \mathbb{R}^d$. Suppose that

(a) the coefficients σ, b are $\mathcal{C}^\infty$ functions with bounded derivatives of any order greater or equal than one,

(b) there exists $\sigma_0 > 0$ such that $\inf\{|\sigma(z)|,\ z \in \mathbb{R}\} \ge \sigma_0$,

(c) there exists $\eta \in (0, \frac{1}{2})$ such that condition (H_η) holds.

Then the law of $u(t, x)$ has an infinite differentiable density with respect to the Lebesgue measure on $\mathbb{R}$.

Proof

We proceed as in the proof of the preceding Theorem 8.1. Notice that the hypotheses 3) of Proposition 8.1 hold with $\theta_1 = 3$, $\theta_2 = 3 - 2\eta$ and $\theta_3 = 2$. These parameters satisfy the restriction $\theta_1 < \frac{3}{2}\theta_2 \wedge (\theta_2 + \theta_3)$ if $\eta \in (0, \frac{1}{2})$. $\square$

Our next purpose is to study the stochastic wave equation for $d = 3$. The fundamental solution of the underlying differential operator is no longer a function, but a non-negative measure. Thus our aim is to apply Proposition 8.2. In addition to the work done in lower dimensions we must analyse the validity of (8.21). The next three lemmas give the technical background.

Lemma 8.7

Suppose that there exists $\eta \in (0, \frac{1}{2})$ such that (H_η) is satisfied. Then for any $t \in [0, T]$,

$$\int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)|\xi| \left|\mathcal{F}\Lambda(s)(\xi)\right|^2 \le Ct^{2-2\eta}, \tag{8.38}$$

with

$$C = \frac{\mu\{|\xi| \le 1\}}{3} T^{1+2\eta} + \frac{1}{(2-2\eta)2^{1+\eta}\pi^{1+2\eta}} \int_{|\xi|>1} \frac{\mu(d\xi)}{\left(1+|\xi|^2\right)^\eta}.$$

Proof

We decompose the left hand-side of (8.38) into the sum $J_1(t) + J_2(t)$, with

$$J_1(t) = \int_0^t ds \int_{|\xi|\le 1} \mu(d\xi)|\xi| \left|\mathcal{F}\Lambda(s)(\xi)\right|^2,$$

$$J_2(t) = \int_0^t ds \int_{|\xi|>1} \mu(d\xi)|\xi| \left|\mathcal{F}\Lambda(s)(\xi)\right|^2.$$

Clearly,

$$J_1(t) \le \mu\{|\xi| \le 1\} \frac{t^3}{3}. \tag{8.39}$$

Let $0 < \gamma < 1$. Then,

$$\begin{aligned} J_2(t) &= \int_0^t ds \int_{|\xi|>1} \mu(d\xi)|\xi| \frac{\left(\sin 2\pi s|\xi|\right)^\gamma}{4\pi^2|\xi|^2} \\ &\le (2\pi)^{\gamma-2} \int_0^t ds \int_{|\xi|>1} \mu(d\xi)|\xi|^{\gamma-1} s^\gamma \end{aligned}$$

$$= (2\pi)^{\gamma-2}\frac{t^{\gamma+1}}{\gamma+1}\int_{|\xi|>1}\frac{\mu(d\xi)}{\big(|\xi|^2\big)^{(1-\gamma)/2}}$$
$$\le \frac{2^{(\gamma-3)/2}\,\pi^{\gamma-2}}{\gamma+1}t^{\gamma+1}\int_{|\xi|>1}\frac{\mu(d\xi)}{\big(1+|\xi|^2\big)^{(1-\gamma)/2}}\,.$$

Let $\eta := \frac{1-\gamma}{2}$. We obtain

$$J_2(t) \le \bar{C}t^{2-2\eta}, \tag{8.40}$$

with

$$\bar{C} = \frac{1}{(2-2\eta)\,2^{1+\eta}\pi^{1+2\eta}}\int_{|\xi|>1}\frac{\mu(d\xi)}{\big(1+|\xi|^2\big)^{\eta}}.$$

Consequently, (8.39) and (8.40) yield (8.38) with the value of the constant C given in the statement. □

Let $\big(Z(t,x),(t,x)\in[0,T]\times\mathbb{R}^d\big)$ be a predictable L^2-process with stationary covariance function such that $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\big(|Z(t,x)|^2\big) < \infty$. We recall the notation $\Gamma_s^Z(dx) = g(s,x)\,\Gamma(dx)$, with $g(s,x) = E\big(Z(s,y)\,Z(s,x+y)\big)$ and $\mu_s^Z = \mathcal{F}^{-1}(\Gamma_s^Z)$.

Set

$$G_{d,\eta}(x) = \mathcal{F}^{-1}\left(\frac{1}{\big(1+|\xi|^2\big)^{\eta}}\right)(x), \qquad d\ge 1,\ \eta\in(0,\infty).$$

It is well-known (see for instance ref. [17]) that

$$G_{d,\eta}(x) = C_{d,\eta}|x|^{\eta-\frac{d}{2}}K_{\frac{d}{2}-\eta}\big(|x|\big),$$

where $C_{d,\eta}$ is some strictly positive constant and K_ρ is the modified *Bessel function* of second kind of order ρ. Set

$$F_{d,\eta}(y) = \int_{\mathbb{R}^d}\Gamma(dx)G_{d,\eta}(x-y),$$

$y\in\mathbb{R}^d$. We remark that if the function $\varphi = 1/\big(1+|\xi|^2\big)^{\eta}$ where in $\mathcal{S}(\mathbb{R}^d)$ — which is not the case — then the following equality would hold:

$$F_{d,\eta}(0) = \int_{\mathbb{R}^d}\frac{\mu(d\xi)}{\big(1+|\xi|^2\big)^{\eta}}\,.$$

The next lemma (cf. Section 4.4 of ref. [32]) clarifies the relation between the property (H_η) and the finiteness of $F_{d,\eta}(y)$.

Lemma 8.8

For any $\eta \in (0,\infty)$ *the following statements are equivalent*

(i) $\sup_{y\in\mathbb{R}^d} F_{d,\eta}(y) = \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \Gamma(dx) G_{d,\eta}(x-y) < \infty,$

(ii) $\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{\left(1+|\xi|^2\right)^\eta} < \infty.$

In fact, if either (i) or (ii) hold, then

$$\sup_{y\in\mathbb{R}^d} F_{d,\eta}(y) = \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{\left(1+|\xi|^2\right)^\eta}.$$

Proof

Assume (i). For any $t>0$ set $p_t = \mathcal{F}^{-1}\left(e^{-2\pi^2 t|\xi|^2}\right)$. Since p_t is the density of a probability measure on $\mathbb{R}^d$ we clearly have that

$$\sup_{t>0} \int_{\mathbb{R}^d} dy p_t(y) F_{d,\eta}(y) \le \sup_{y\in\mathbb{R}^d} F_{d,\eta}(y) < \infty.$$

The definition of $F_{d,\eta}$ and Fubini's Theorem yields

$$\int_{\mathbb{R}^d} p_t(y) F_{d,\eta}(y) = \int_{\mathbb{R}^d} \Gamma(dx)(G_{d,\eta} * p_t)(x).$$

Since $G_{d,\eta} * p_t \in \mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \Gamma(dx)(G_{d,\eta} * p_t)(x) = \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{\left(1+|\xi|^2\right)^\eta}\, e^{-2\pi^2 t|\xi|^2}.$$

By monotone convergence,

$$\lim_{t\to 0} \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{\left(1+|\xi|^2\right)^\eta}\, e^{-2\pi^2 t|\xi|^2} = \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{\left(1+|\xi|^2\right)^\eta},$$

thus,

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{\left(1+|\xi|^2\right)^\eta} \le \sup_{y\in\mathbb{R}^d} F_{d,\eta}(y) < \infty,$$

proving (ii).

Assume now that (ii) holds. Then, the Fourier transform of the finite measure $\mu(d\xi)/\left(1+|\xi|^2\right)^\eta$ is a bounded function, which is also the convolution of the Fourier transforms of $\mu(d\xi)$ and $1/\left(1+|\xi|^2\right)^\eta$, that is, $F_{d,\eta}$. In particular, $F_{d,\eta}$ is bounded. □

The next lemma is a technical result needed in the proof of the analogue of Lemma 8.7 for the measure μ_s^Z.

Lemma 8.9

Assume that (H_η) *holds for some* $\eta \in (0,1)$. *Then*

$$\sup_{0\leq s\leq T} \int_{\mathbb{R}^d} \frac{\mu_s^Z(d\xi)}{\left(1+|\xi|^2\right)^\eta} \leq C \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{\left(1+|\xi|^2\right)^\eta},$$

for some positive constant C.

Proof

Set

$$F_{d,\eta}^Z(s,y) := \int_{\mathbb{R}^d} \Gamma_s^Z(dx) G_{d,\eta}(x-y), \qquad s\in[0,T],\ y\in\mathbb{R}^d.$$

Lemma 8.8 implies that

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}^d} F_{d,\eta}^Z(s,y) < \infty.$$

Indeed, this follows from the definition of the measure Γ_s^Z and the properties of the process Z.

Then, again by Lemma 8.8, it follows that for any $s\in[0,T]$

$$\int_{\mathbb{R}^d} \frac{\mu_s^Z(d\xi)}{\left(1+|\xi|^2\right)^\eta} < \infty.$$

Let p_t be as in the preceding lemma; by bounded convergence we have

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{\mu_s^Z(d\xi)}{\left(1+|\xi|^2\right)^\eta} &= \lim_{t\searrow 0} \int_{\mathbb{R}^d} \mu_s^Z(d\xi) \frac{\exp^{-2\pi^2 t|\xi|^2}}{\left(1+|\xi|^2\right)^\eta} \\ &= \lim_{t\searrow 0} \int_{\mathbb{R}^d} \Gamma_s^Z(dx)(G_{d,\eta} * p_t)(x).\end{aligned}$$

Fubini's Theorem yields that

$$\int_{\mathbb{R}^d} \Gamma_s^Z(dx)(G_{d,\eta} * p_t)(x) = \int_{\mathbb{R}^d} dy\, p_t(y) F_{d,\eta}^Z(s,y).$$

But, the definition of Γ_s^Z implies

$$\begin{aligned}
&\int_{\mathbb{R}^d} dy\, p_t(y) F_{d,\eta}^Z(s,y)\\
&\quad= \int_{\mathbb{R}^d} dy\, p_t(y) \int_{\mathbb{R}^d} \Gamma_s^Z(dx) G_{d,\eta}(x-y)\\
&\quad= \int_{\mathbb{R}^d} dy\, p_t(y) \int_{\mathbb{R}^d} \Gamma(dx) g(s,x) G_{d,\eta}(x-y)\\
&\quad\le \sup_{(s,x)\in[0,T]\times\mathbb{R}^d} E\big(|Z(s,x)|^2\big) \int_{\mathbb{R}^d} dy\; p_t(y) \int_{\mathbb{R}^d} \Gamma(dx) G_{d,\eta}(x-y)\\
&\quad= C \int_{\mathbb{R}^d} \Gamma(dx)(G_{d,\eta} * p_t)(x) = C \int_{\mathbb{R}^d} \mu(d\xi) \frac{\exp^{-2\pi^2 t|\xi|^2}}{\big(1+|\xi|^2\big)^\eta}.
\end{aligned}$$

Owing to (H_η) and using again bounded convergence, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^d} \mu_s^Z(d\xi) \frac{1}{\big(1+|\xi|^2\big)^\eta} &\le C \lim_{t\searrow 0} \int_{\mathbb{R}^d} \mu(d\xi) \frac{\exp^{-2\pi^2 t|\xi|^2}}{\big(1+|\xi|^2\big)^\eta}\\
&= C \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{\big(1+|\xi|^2\big)^\eta}.
\end{aligned}$$

□

We can now give the last ingredient we need.

Lemma 8.10

Assume that (H_η) holds with η restricted to the interval $(0, \frac{1}{2})$. Then, for any $t \in [0,T]$ there exists a positive constant C such that

$$\int_0^t ds \int_{\mathbb{R}^d} \mu_s^Z(d\xi)|\xi| \left|\mathcal{F}\Lambda(s)(\xi)\right|^2 \le C t^{2-2\eta}. \tag{8.41}$$

Proof

Clearly, by the inequality (6.8) with $p = 2$ and Lemma 8.6 (see (8.36)),

$$T_1(t) := \int_0^t ds \int_{\{|\xi|\le 1\}} \mu_s^Z(d\xi)|\xi| \left|\mathcal{F}\Lambda(s)(\xi)\right|^2 \le C t^{3-2\eta}. \tag{8.42}$$

Using the same arguments as those in the proof of Lemma 8.7 to study the term $J_2(t)$, we obtain that

$$\begin{aligned} T_2(t) &:= \int_0^t ds \int_{\{|\xi|>1\}} \mu_s^Z(d\xi)|\xi| \frac{\sin^2\big(2\pi s|\xi|\big)}{4\pi^2|\xi|^2} \\ &\le \int_0^t ds \int_{\{|\xi|>1\}} \mu_s^Z(d\xi)|\xi| \frac{\Big(\sin(2\pi s|\xi|)\Big)^{1-2\eta}}{4\pi^2|\xi|^2} \\ &\le C \int_0^t ds\, s^{1-2\eta} \int_{\mathbb{R}^d} \mu_s^Z(d\xi) \frac{1}{\big(1+|\xi|^2\big)^{\eta}}. \end{aligned}$$

Due to the preceding lemma, this last term is bounded by $Ct^{2-2\eta}$, which together with (8.42) imply (8.41). □

We can now give the result on existence and smoothness of density for the stochastic wave equation in dimension $d = 3$. The restriction on the dimension is imposed by the non-negative requirement on the fundamental solution in order to have existence and uniqueness of a real-valued solution to Equation (6.13) (see Theorem 6.2 and Example 6.1).

Theorem 8.3

Let $u(t,x)$ be the solution of Equation (6.13) with $L := \partial_{tt}^2 - \triangle_3$ at a fixed point $(t,x) \in (0,T] \times \mathbb{R}^3$. Suppose that

(a) the coefficients σ, b are $\mathcal{C}^\infty$ functions with bounded derivatives of any order greater or equal than one,

(b) there exists $\sigma_0 > 0$ such that $\inf\big\{|\sigma(z)|,\ z \in \mathbb{R}\big\} \ge \sigma_0$,

(c) there exists $\eta \in (0, \frac{1}{2})$ such that condition (H_η) holds.

Then the law of $u(t,x)$ has an infinite differentiable density with respect to the Lebesgue measure on $\mathbb{R}$.

Proof

We apply Theorem 5.2. Assumption (a) of this theorem is assured by Theorem 7.1 We next prove that the hypotheses of Proposition 8.2 are satisfied. Thus, condition (b) of the above mentioned Theorem 5.2 also holds true. Indeed, by Lemma 8.7 the upper bound (8.21) holds with $\theta_4 = 2-2\eta$. Applying Lemma 8.10 to the process $Z(s,x) = \sigma\big(u(t-s,x)\big)$ yields that the upper bound (8.22) is satisfied with $\theta_5 = 2-2\eta$. On the other hand we already know that (8.19) and (8.20) hold with $\theta_1 = 3$, $\theta_2 = 3-2\eta$, $\theta_3 = 2$. Then, $\theta_4 = \theta_5 < \theta_1 < \frac{3}{2}\theta_2 \wedge (\theta_2 + \theta_3)$. □

8.3 Multidimensional case

Let $x_1, \dots, x_m$ be distinct points of $\mathbb{R}^d$. Consider the solution of Equation (6.13) at $(t, x_1), \dots, (t, x_m)$. Set $u(t, \underline{x}) = \big(u(t, x_1), \dots, u(t, x_m)\big)$. We denote by $\Gamma(t, \underline{x})$ the Malliavin matrix of $u(t, \underline{x})$, that is,

$$\Big(\big\langle Du(t, x_i), Du(t, x_j)\big\rangle_{\mathcal{H}_T},\ 1 \le i,\ j \le m\Big).$$

We assume that $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\big(\|Du(t,x)\|^p_{\mathcal{H}_T}\big) < \infty$, for any $p \in [1, \infty)$ (see Theorem 7.1).

In this section we study sufficient conditions ensuring the property

Property (J) *For every* $p > 0$,

$$E\big(\det \Gamma(t, \underline{x})^{-p}\big) < \infty. \tag{8.43}$$

We start with a result which has an analogous function as Lemma 8.3 in our new context.

Lemma 8.11

Fix $p > 0$. *Assume that for any* $v \in \mathbb{R}^m$ *there exists* $\epsilon_0 > 0$, *depending on* p *and* v *such that*

$$\int_0^{\epsilon_0} \epsilon^{-(1+pm+2m)} P\big(v^T \Gamma(t, \underline{x}) v < 2\epsilon\big) < \infty. \tag{8.44}$$

Then, (8.43) holds true.

Proof

Let $\lambda(t, \underline{x}) = \inf_{|v|=1} v^T \Gamma(t, \underline{x}) v$. Then $\det \Gamma(t, \underline{x}) \ge \big(\lambda(t, \underline{x})\big)^m$. Set $q = pm$; it suffices to check that

$$E\big(\lambda(t, \underline{x})\big)^{-q} < \infty.$$

A simple argument yields the following (see, for instance Lemma 2.3.1 in ref. [43]): For any $\epsilon > 0$,

$$P\big(\lambda(t, \underline{x}) < \epsilon\big) \le \sum_{k=1}^{n_0} P\Big(v_k^T \Gamma(t, \underline{x}) v_k < 2\epsilon\Big) + P\Big(\big\|\Gamma(t, \underline{x})\big\| > \epsilon^{-1}\Big), \tag{8.45}$$

where n_0 denotes the number of balls centered at the unit vectors of $\mathbb{R}^m$, $v_1, \dots, v_{n_0}$ with radius $\frac{\epsilon^2}{2}$ covering the unit sphere S^{m-1}, and $\|\cdot\|$ denotes the Hilbert-Schmidt norm. Notice that $n_0 \le C\epsilon^{-2m}$.

Set $F = \big(\lambda(t,\underline{x})\big)^{-q}$. The classical argument used in the proof of Lemma 8.3 yields

$$E(F) \le \eta_0 + q \int_0^{\eta_0^{-1/q}} \epsilon^{-(q+1)} P\big(\lambda(t,\underline{x}) < \epsilon\big)\, d\epsilon. \tag{8.46}$$

Since $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\big(\|Du(t,x)\|^p_{\mathcal{H}_T}\big) < \infty$, for any $p \in [1,\infty)$, Chebychev's inequality yields

$$P\Big(\big\|\Gamma(t,\underline{x})\big\| > \epsilon^{-1}\Big) \le \epsilon^r E\Big(\big\|\Gamma(t,\underline{x})\big\|^r\Big) \le C\epsilon^r,$$

for any $r \in [1,\infty)$. Therefore, in view of (8.45) and (8.46) we conclude. □

We would like now to carry out a similar program as in Section 8.1. However, the results are still not satisfactory. In spite of this feeling, we present a general result in the next proposition. It is the multidimensional version of Proposition 8.1.

This result is applied to the wave equation in spatial dimension $d = 2$ when the correlation of the noise is given by a *Riesz kernel*; this is the main topic of reference [40].

We hope to be able to study the stochastic heat equation with the same tools.

Similarly, we could prove the multidimensional analogue of Proposition 8.2. However we do not yet have any examples where it could be applied.

Proposition 8.4

Suppose that Hypothesis D is satisfied and in addition that the measure $\Lambda(t)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. Moreover, assume that

1) *the coefficients σ and b are $\mathcal{C}^1$ functions with bounded Lipschitz continuous derivatives,*
2) *there exists $\sigma_0 > 0$ such that* $\inf\big\{|\sigma(z)|,\ z \in \mathbb{R}\big\} \ge \sigma_0$,

3) there exist θ_i, $C_i > 0$, $i = 1,2,3,4$, satisfying $\theta_1 < \frac{3}{2}\theta_2 \wedge (\theta_2 + \theta_3) \wedge \theta_4$, such that for any $t \in (0,1)$,

$$C_1 t^{\theta_1} \le \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) \left|\mathcal{F}\Lambda(s)(\xi)\right|^2 \le C_2 t^{\theta_2}, \tag{8.47}$$

$$\int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,y) dy \le C_3 t^{\theta_3}, \tag{8.48}$$

$$\int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) \left|\mathcal{F}\Lambda(s)(x_1 - \cdot)\right| \left|\overline{\mathcal{F}\Lambda(s)(x_2 - \cdot)}\right| \le C_4 t^{\theta_4}, \tag{8.49}$$

for any x_1, x_2 different points in $\mathbb{R}^d$

Then (J) holds.

Proof

Let $\left(\xi_{r,z}(t,x),\, (t,x) \in [0,T] \times \mathbb{R}^d,\, r \le t,\, z \in \mathbb{R}^d\right)$, be the solution of the equation

$$\begin{aligned} \xi_{r,z}(t,x) = \Lambda(t-r, x-z) &+ \int_r^t \int_{\mathbb{R}^d} \Lambda(t-s, x-y) \\ &\quad + \sigma'\big(u(s,y)\big)\xi_{r,z}(s,y) M(ds,dy) \\ &+ \int_r^t \int_{\mathbb{R}^d} \Lambda(s,dy) \\ &\quad + b'\big(u(t-s,x-y)\big)\xi_{r,z}(t-s,x-y), \end{aligned} \tag{8.50}$$

for fixed r,z. By uniqueness of solution $D_{r,z}u(t,x) = \sigma\big(u(r,z)\big)\xi_{r,z}(t,x)$. Let $\epsilon_1, \delta > 0$ be such that $t - \epsilon^\delta > 0$ for any $0 \le \epsilon \le \epsilon_1$. Then, if $v = (v_1, \ldots, v_m)$, by hypothesis 2),

$$v^T \Gamma(t,\underline{x}) v \ge \sigma_0^2 \sum_{i,j=1}^m \int_{t-\epsilon^\delta}^t dr \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy\, v_i v_j \xi_{r,y}(t,x_i) \xi_{r,x-y}(t,x_j).$$

Therefore, by the triangular inequality,

$$P(v^T \Gamma(t,\underline{x}) v < \epsilon) \le p^1(\epsilon,\delta) + p^2(\epsilon,\delta), \tag{8.51}$$

where

$$p^1(\epsilon,\delta) = P\left(\int_{t-\epsilon^\delta}^t dr \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy\, v_j^2 \xi_{r,y}(t,x_j)\xi_{r,x-y}(t,x_j) < \frac{2}{\sigma_0^2}\epsilon\right),$$

$$p^2(\epsilon,\delta) = P\left(\sum_{i\neq j} \int_{t-\epsilon^\delta}^t dr \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy\, v_i v_j \xi_{r,y}(t,x_i) \times \xi_{r,x-y}(t,x_j) \geq \frac{\epsilon}{\sigma_0^2}\right),$$

for any $j = 1, 2, \ldots, m$.

We study the ϵ-size of the term $p^1(\epsilon,\delta)$ following the same arguments as in the proof of Proposition 8.1. We come out with the following conclusion: Fix $p \in [1,\infty)$. Assume there exist $q \in [1,\infty)$ and $\delta > 0$ such that

$$\theta_1 < \frac{1}{\delta} < \frac{\left(\frac{3}{2}\theta_2 \wedge (\theta_2 + \theta_3)\right)q}{pm + 2m + q}. \tag{8.52}$$

Then the function $\varphi(\epsilon) = \epsilon^{-(1+pm+2m)} p^1(\epsilon,\delta)$ is integrable in a neighbourhood of zero.

Chebychev's inequality yields

$$p^2(\epsilon,\delta) \leq C\epsilon^{-q} \sup_{i\neq j} E\left(\left|\langle \xi_{t-\cdot,*}(t,x_i), \xi_{t-\cdot,*}(t,x_j)\rangle_{\mathcal{H}_{\epsilon^\delta}}\right|^q\right). \tag{8.53}$$

By virtue of the equation (8.50), and following similar arguments as those of the proof of Lemma 8.2 one can check that the right hand-side of (8.53) is bounded by

$$C \sup_{i\neq j} \left(\int_0^{\epsilon^\delta} ds \int_{\mathbb{R}^d} \mu(d\xi) \left|\mathcal{F}\Lambda(s)(x_i - \cdot)\right| \left|\overline{\mathcal{F}\Lambda(s)(x_j - \cdot)}\right|\right)^q.$$

Hence, owing to (8.49)

$$p^2(\epsilon,\delta) \leq C\epsilon^{q(-1+\delta\theta_4)}.$$

Consequently the integrability of the function

$$\psi(\epsilon) = \epsilon^{-(1+pm+2m)} p^2(\epsilon,\delta)$$

in a neighbourhood of zero is assured as far as

$$\frac{1}{\delta} < \frac{q\theta_4}{pm + 2m + q}, \tag{8.54}$$

for some $q \in [1,\infty)$.

We conclude by checking that both restrictions (8.52) and (8.54) are compatible under the assumptions on θ_i, $i = 1, \dots, 4$ given in the statement. □

Examples 8.1

Consider the stochastic wave equation in dimension $d = 2$. We assume that $\Gamma(dx) = f(x)dx$, with $f(x) = |x|^{-\alpha}$ with $\alpha \in (0, 2)$ and the same assumptions (a) and (b) of Theorem 8.2. Then the law of the random vector $u(t, \underline{x})$ has an infinite differentiable density with respect to the Lebesgue measure on $\mathbb{R}^m$.

Indeed, let us check that the assumptions of Theorem 5.2 are satisfied. Hypothesis (a) follows from Proposition 7.1, while condition (b) shall follow from the previous proposition.

In fact, Lemma A1 in reference [40] states that

$$\int_0^t ds \int_{\mathbb{R}^2} \mu(d\xi) \big|\mathcal{F}\Lambda(s)(\xi)\big|^2 \sim t \int_0^t r f(r) \ln\left(1 + \frac{t}{r}\right) dr,$$

by virtue of the particular expression of the fundamental solution.

On the other hand, by the particular choice of the correlation density, it is easy to check that

$$\int_0^t r f(r) \ln\left(1 + \frac{t}{r}\right) dr \sim t^{2-\alpha}. \tag{8.55}$$

Consequently, $\theta_1 = \theta_2 = 3 - \alpha$. We already know that $\theta_3 = 2$. Let us now prove that $\theta_4 = 3$.

Set $m = |x_1 - x_2|$. Then, if $4t < m$, $|z - x_1| < t$, $|z' - x_2| < t$ imply $m/2 \le |z - z'| \le (3m)/2$. Hence, since f is continuous, for these range of $z, z' \in \mathbb{R}^2$, $f\big(|z - z'|\big)$ is bounded.

Therefore,

$$\begin{aligned}
&\int_0^t ds \int_{\mathbb{R}^2} \mu(d\xi) \big|\mathcal{F}\Lambda(s)(x_1 - \cdot)\big| \left|\overline{\mathcal{F}\Lambda(s)(x_2 - \cdot)}\right| \\
&\qquad = \int_0^t ds \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} dx\, dz f(x) \Lambda(s, z - x) \Lambda(s, z) \\
&\qquad = \int_0^t ds \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(z - z') \Lambda(s, z') \Lambda(s, z) \\
&\qquad \le C \int_0^t \left(\int_{(|z| \le s)} \frac{dz}{\sqrt{s^2 - |z|^2}} \right)^2 ds \le C t^3,
\end{aligned}$$

where we made the change of variables $z - x = z'$.

It is trivial to check that for any $\alpha \in (0, 2)$ these values of θ_i, $i = 1, \dots, 4$, satisfy the conditions of Proposition 8.4.

Remark 8.2 It is natural to compare the assumptions on Γ in the preceding example with the validity of condition (H_η). We prove now that if $\alpha \in (0, 2\eta)$, then (H_η) holds true.

Indeed, it is well known that $\mathcal{F}f(\xi) = |\xi|^{-(2-\alpha)}$. Then condition (H_η) is equivalent to

$$\int_0^\infty \frac{\rho\, d\rho}{\rho^{2-\alpha}\big(1+\rho^2\big)^\eta} < \infty.$$

Clearly,

$$\int_0^1 \frac{\rho\, d\rho}{\rho^{2-\alpha}\big(1+\rho^2\big)^\eta} \le \int_0^1 \rho^{-1+\alpha}\, d\rho < \infty,$$

because $\alpha > 0$. Moreover

$$\int_1^\infty \frac{\rho\, d\rho}{\rho^{2-\alpha}\big(1+\rho^2\big)^\eta} \le \int_1^\infty \rho^{-1+\alpha-2\eta}\, d\rho < \infty,$$

since $\alpha < 2\eta$.

Remark 8.3 There are two facts in Example 8.1 worthy of being mentioned. The first one is that the value $\theta_1 = 3-\alpha$ is better that the one obtained in (8.35) ($\theta_1 = 3$), which means that the results of Lemma A1 in reference [40] are sharper than those of Lemma 8.6 for $d = 2$. The second one is the value of θ_4 which has been obtained using the particular form of the fundamental solution in this dimension. One could try to apply Schwarz's inequality to

$$\int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) \big|\mathcal{F}\Lambda(s)(x_1-\cdot)\big|\,\Big|\overline{\mathcal{F}\Lambda(s)(x_2-\cdot)}\Big|$$

and then use the bound of $\int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) \big|\mathcal{F}\Lambda(s)(x-\cdot)\big|^2$. This procedure gives a rougher inequality ($\theta_2 = \theta_4$) which is not suitable for the analysis of Example 8.1.

Remark 8.4 In view of the preceding remarks it seems that an extension of Theorem 8.3 to the multidimensional case requires, as in dimension 2, a strengthening of Lemma 8.6. Due to the complexity of the above mentioned Lemma A1 in reference [40] this seems to be a difficult and challenging problem.

Comments

In this chapter we have followed the strategy of reference [35] of giving sufficient conditions on the behaviour of the fundamental solution ensuring non-degeneracy of the Malliavin matrix. In comparison with reference [35] our results apply to a broader class of equations including the stochastic wave equation with spatial dimension of three. Proposition 8.2 is an abstract formulation of results published in reference [55]. Section 8.2 contains results from references [32], [35] and [55]. Lemma 8.8 appears essentially in reference [32].

We believe that the results concerning the stochastic wave equation can be extended to the damped wave equation using the analysis of the fundamental solution carried out in reference [32].

Definitions of spaces

$\mathcal{C}_0^r(\mathbb{R}^m)$, $r \in (0, \infty]$	space of r-times differentiable functions with compact support.
$\mathcal{C}_b^r(\mathbb{R}^m)$, $r \in (0, \infty]$	space of bounded, r-times differentiable functions with bounded derivatives up to order r.
$\mathcal{C}^\infty(\mathbb{R}^m)$	space of infinitely differentiable functions defined on $\mathbb{R}^m$. For $m = 1$, we write $\mathcal{C}^\infty$ instead of $\mathcal{C}^\infty(\mathbb{R})$.
$\mathcal{C}_p^\infty(\mathbb{R}^m)$	space of infinitely differentiable functions f defined on $\mathbb{R}^m$ such that f and its partial derivatives have polynomial growth.
$\mathcal{D}(\mathbb{R}^m)$	space of Schwartz test functions, that is, the topological vector space of functions in $\mathcal{C}_0^\infty(\mathbb{R}^m)$ with the topology induced by the following notion of convergence: $\varphi_n \to \varphi$ if and only if 1) there is a compact subset $K \subset \mathbb{R}^m$ such that $\operatorname{supp}(\varphi_n - \varphi) \subset K$, for all n; 2) $\lim_{n\to\infty} \nabla^\alpha \varphi_n = \nabla^\alpha \varphi$, uniformly on K, for any multi-index α.
$\mathcal{S}(\mathbb{R}^m)$	space of $\mathcal{C}^\infty(\mathbb{R}^m)$ functions with rapid decrease.
$\mathcal{B}_b(\mathbb{R}^m)$	set of Borel bounded subsets of $\mathbb{R}^m$.
$\mathcal{P}$	set of Gaussian functionals of the form $f\big(W(h_1), \dots, W(h_n)\big)$, where f is a polynomial.
$\mathcal{S}$	set of Gaussian functionals of the form $f\big(W(h_1), \dots, W(h_n)\big)$, where $f \in \mathcal{C}_p^\infty(\mathbb{R}^m)$.
$\mathcal{S}_b$	set of Gaussian functionals of the form $f\big(W(h_1), \cdots, W(h_n)\big)$, where $f \in \mathcal{C}_b^\infty(\mathbb{R}^m)$.
$\mathcal{S}_\mathcal{H}$	set of random vectors of the form $u = \sum_{j=1}^n F_j h_j$, $F_j \in \mathcal{S}$, $h_j \in H$, $j = 1, \dots, n$.
$\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$	set of linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$.

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